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LUMPED-PARAMETER AND FINITE-ELEMENT-MODELS FOR DYNAMIC BEHAVIOR OF PLANE STRAIGHT SIDED FRAMES

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LUMPED-PARAMETER AND FINITE-ELEMENT-MODELS FOR DYNAMIC BEHAVIOR OF PLANE STRAIGHT SIDED FRAMES

by

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ABSTRACT

Lumped-parameter and finite-element-models are developed for the dynamic behavior of plane straight sided frames. The models do not include axial and shear deformation, but the equations of motion developed using the models allow for time varying external loading. The performance of these models is evaluated by a comparison with a standard transfer matrix method for the special case of free undamped vibration. The finite-element-model proves to be much the better model. For the first five modes, the finite-element-model with 27 degrees of freedom differs by no more than 0.4% from the "exact values" given by the transfer matrix method. The lumped-parameter-model with 39 degrees of freedom gives errors roughly ten times as great.

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NOMENCLAUTRE

A_i = Cross sectional area of member i.

$\underline{B}$ = Spectral matrix from diagonalization of $\underline{J}$.

$\underline{C}$ = Matrix relating $\underline{d}$ to $\underline{c}$.

$\underline{c}$ = Coefficient vector.

c_i = Element of coefficient vector

$\underline{D}$ = Modal matrix from diagonalization of $\underline{J}$.

$\underline{d}$ = Deflection and slope vector.

E_i = Modulus of elasticity of member i.

EI = Flexural rigidity

$\underline{f}$ = External force vector.

$\underline{f}_x$ = Vector of x components of external force.

f_{xj} = x component of external force at node j.

$\underline{f}_y$ = Vector of y components of external force.

f_{yj} = y component of external force at node j.

$\underline{g}^t$ = Row vector in powers of r for $\frac{d^2w}{dr^2}$.

$\underline{h}^t$ = Row vector in powers of r for ϕ.

$\underline{I}$ = Identity matrix.

I_i = Second moment of area A_i with respect to a centroidal axis perpendicular to plane of frame.

$\underline{J}$ = Matrix sum of inertia terms.

$\underline{J}_o$ = Inertia matrix.

J_j = Mass moment of inertia of segment j about an axis normal to the plane of the structure and thru the segment centroid.

J_{jj} = Rotational inertia between j and j+1.

$\underline{K}$ = Stiffness matrix.

k_j = Stiffness of segment j.

$\underline{L}$ = Incidence matrix.

ℓ_i = Length of segment of member i, = $\frac{s_i}{n_i}$.

$\underline{M}$ = Mass matrix.

$\underline{M}_r$ = Rotational mass matrix.

$\underline{M}_t$ = Translational mass matrix.

m_i = Segment mass.

$\underline{\mathfrak{m}}$ = Moment vector.

$\mathfrak{m}_j$ = Moment applied between node j and j-1.

N = Total number of subdivisions.

n_i = Number of subdivisions in member i.

$\underline{P}$ = Matrix used to apply end conditions in lumped-parameter-model.

$\underline{p}^t$ = Row matrix in powers of r for w.

$\underline{Q}$ = Moment arm matrix for external force vector.

$\underline{q}$ = Generalized force vector.

$\underline{R}$ = Corner transformation matrix.

r = Axial coordinate.

$\underline{S}$ = Matrix product of $\underline{B}^{-\frac{1}{2}}\underline{D}^t\underline{K}\,\underline{D}\underline{B}^{-\frac{1}{2}}$.

s_i = Length of member i.

T = Kinetic energy.

T_r = Rotational kinetic energy.

T_t = Translational kinetic energy.

U = Potential energy.

$\underline{u}$ = Vector of x components of displacement

u = Axial displacement.

u_j = x displacement of node j.

$\underline{v}$ = Vector of y components of displacement.

v_j = y displacement of node j.

W_i = Specific weight of member i.

w = Transverse deflection.

$\underline{X}$ = Matrix of the x components of distance.

X_j = x-coordinate of the centroid of segment j.

x_{ij} = x component of distance from node j to node i+1.

$\underline{Y}$ = Matrix of the y components of distance.

Y_j = y-coordinate of the centroid of segment j.

y_{ij} = y component of distance from node j to node i+1.

$\underline{z}$ = product of $\underline{B}^{\frac{1}{2}}$ and $\underline{D}^{t}$.

(x,y,z) = Cartesian coordinate system.

α = Deviation angle at a corner of the frame.

β = cos α.

γ = Angle between axial direction in local coordinate system and x direction in Cartesian system.

η = sin α.

$\underline{\theta}$ = Relative rotation vector.

θ_j = Relative rotation of hinge j.

μ = Mass per unit length.

ϕ = Slope.

ω = Natural circular frequency.

Subscripts

a = assembled matrices.

i = Property of member i.

j = Property at node j.

ij = Property from node i to node j in finite-element-model; from node j to node i+1 in lumped-parameter-model.

u = Upper rows of a matrix as determined by end conditions.

ℓ = Remaining rows of matrix.

Superscripts

t = Transpose of vectors and matrices.

" = Matrix after application of end conditions.

* = Vector resulting from corner transformation.

' = In lumped-parameter-model denotes a once reduced matrix; in finite-element-model denotes the properties of the element to the right of the corner.

INTRODUCTION

When designing a structure, the engineer must be aware of the many dynamic disturbances that may act upon this structure. He must also be aware of the response that the structure will have to these disturbances. The calculation of the response is usually a very difficult undertaking. To assist in this calculation, the structure is generally characterized by a simplified model, and the analysis carried out using this model.

In this paper, two models for analyzing plane frame structures with "chain" topology* are developed. The first model lumps both mass and flexibility and the second is a finite-element-model. To test these models, analyses to find the in-plane circular frequencies and translational displacements are carried out on a typical gable bent. The results of these analyses are compared to frequencies and displacements obtained through the use of the so-called transfer matrix method.[1]**

The equations of motion as developed for the models provide for time varying external loads, but do not include damping. The analysis of the gable bent is carried out for the special case of free vibration. This simplification is required so that the principal modes obtained through the use of the models may be compared with values from the transfer matrix method.

---

*Plane framed structures having 2 ends and a single path from end to end.

**Superscripts refer to the bibliography which appears on page 48.

DEVELOPMENT OF THE LUMPED-PARAMETER-MODEL

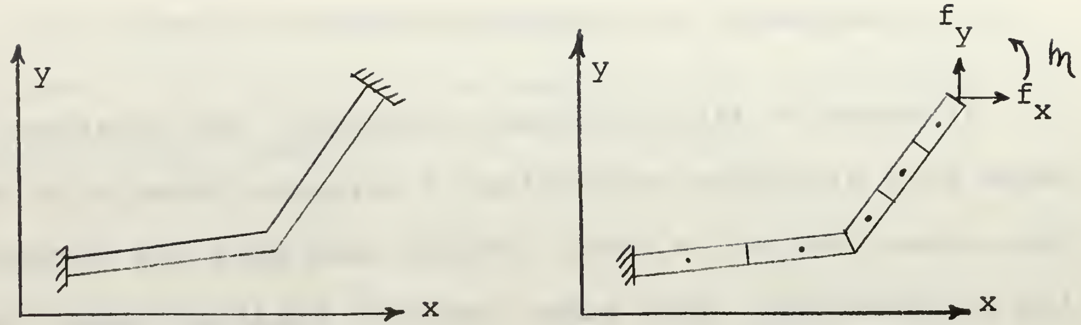
In order to find in-plane frequencies and displacements of a structure consisting of straight sides with uniform cross section, a model lumping both mass and flexibility is developed. This model neglects axial and shear deformation, but can include rotatory inertia.

The model is formed by subdividing each straight member into segments. These segments are considered to have square ends. The length along the centerline of the straight member and the sum of the lengths of the segments replacing it are the same, even though this would cause some overlapping at the corners. The mass and flexibility of each segment is lumped at the segment centroid. The structure is then replaced by a chain of straight rigid bars, pin connected at the locations of the segment centroids. The mass of the segment remains concentrated at the hinge, and the flexibility of the segment is replaced by a torsional spring. The equations of motion are developed and the model analyzed to obtain its natural frequencies and the displacements of the hinges.

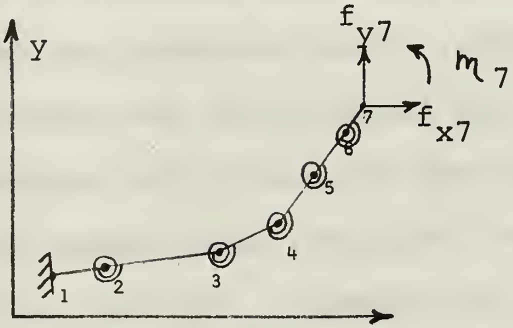
Rigid body translation is prevented by taking the left end as fixed or pinned. The right end is allowed to assume any of the common end conditions: fixed, pinned, simply supported, or free.

Development of the Equations of Motion

To illustrate the method, a structure containing two straight members intersecting at an acute angle is used.



(a) Two Member Frame (b) Segmented Frame



(c) Lumped Model of Frame

Figure 1

Both ends of the structure are taken as fixed (Figure 1a). For each straight member i (i=1,2), let

W_i = specific weight, lb/cu.in.

A_i = cross sectional area, sq.in.

I_i = second moment of the cross sectional area with respect to the z axis, $in.^4$

E_i = modulus of elasticity, lb/sq.in.

s_i = length of member, in.

n_i = number of segments; $n_1=2$, $n_2=3$.

Each section of the structure is divided into n_i segments (Figure 1b). The n_i need not be the same for the different members, but in each member the segments are of uniform length. The parameters of each segment are then lumped at its centroid. These parameters are

ℓ_i = segment length = s_i/n_i inches.

m_i = segment mass = $(AW\ell)_i/386$ lb-sec.^{2}/in.

J_i = mass moment of inertia about an axis normal to the plane of the structure and through the segment centroid,

= $m_i(\ell_i^2/12 + I_i/A_i)$ in-lb-sec.^{2}

k_i = segment stiffness = $(EI)_i/\ell_i$ lb-in.

The structure is then replaced by the rigid bars and torsional springs (Figure 1c). The springs are unstrained in the datum position. The Cartesian coordinates of the end points and hinges of the structure in the datum position are taken as X_j and Y_j where j=1,2,3,4,5,6,7. Thus the displacements of the hinges (u_j, v_j) in the positive x and y directions for a relative rotation, θ_j, at hinge j are given by

$$\begin{aligned} \underline{u} &= -\underline{Y}\ \underline{\theta} \\ \underline{v} &= \underline{X}\ \underline{\theta} \end{aligned} \qquad (1)$$

where

$$\underline{u} = \mathrm{col}(u_2, u_3, u_4, u_5, u_6, u_7)$$

$$\underline{v} = \mathrm{col}(v_2, v_3, v_4, v_5, v_6, v_7)$$

$$\underline{\theta} = \mathrm{col}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$$

$$\underline{Y} = \begin{bmatrix} y_{11} & 0 & 0 & 0 & 0 & 0 \\ y_{21} & y_{22} & 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & y_{33} & 0 & 0 & 0 \\ y_{41} & y_{42} & y_{43} & y_{44} & 0 & 0 \\ y_{51} & y_{52} & y_{53} & y_{54} & y_{55} & 0 \\ y_{61} & y_{62} & y_{63} & y_{64} & y_{65} & y_{66} \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} & 0 & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & 0 \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} \end{bmatrix}$$

$$y_{ij} = Y_{i+1} - Y_j$$

$$x_{ij} = X_{i+1} - X_j$$

The following notation has been used in this equation:

1. Column vectors are denoted by lower case symbols.
2. Rectangular matrices are denoted by capitalized symbols.
3. All vectors and matrices are underlined.

This notation will be used in all other such equations.

Associated with each bar is a rotational inertia made up of half the mass moments of inertia of the preceding and following segments. This rotational inertia is represented by

$$J_{jj} = \frac{1}{2}(J_j + J_{j+1}).$$

Using equation (1), the kinetic energy, T, and the potential energy, U, of the structure are formed.

$$\begin{aligned} T &= \frac{1}{2}\underline{\dot{\theta}}^t\underline{X}^t\underline{MX\dot{\theta}} + \frac{1}{2}\underline{\dot{\theta}}^t\underline{Y}^t\underline{MY\dot{\theta}} + \frac{1}{2}\underline{\dot{\theta}}^t\underline{L}^t\underline{J_o}\underline{L\dot{\theta}} \\ U &= \frac{1}{2}\underline{\theta}^t\underline{K\theta} \end{aligned} \qquad (2)$$

where

$$\underline{M} = \mathrm{diag}(m_2,\ m_3,\ m_4,\ m_5,\ m_6,\ 0)$$

$$\underline{J_o} = \mathrm{diag}(J_{11},\ J_{22},\ J_{33},\ J_{44},\ J_{55},\ J_{66})$$

$$\underline{K} = \mathrm{diag}(k_1,\ k_2,\ k_3,\ k_4,\ k_5,\ k_6)$$

$$\underline{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Derivatives with respect to time are denoted by dots and the transpose of a matrix by a superscript t.

The Lagrangian equations of motion,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{\theta}}}\right) - \frac{\partial T}{\partial \underline{\theta}} + \frac{\partial U}{\partial \underline{\theta}} = \underline{q}$$

may be written

$$\underline{J\ddot{\theta}} + \underline{K\theta} = \underline{q} \qquad (3)$$

where

$$J = \underline{X}^t\underline{MX} + \underline{Y}^t\underline{MY} + \underline{L}^t\underline{J_o}\underline{L}$$

$\underline{q}$ = generalized force vector.

Let the x components of the external forces applied at hinges 2 to 7 be represented by the vector

$$\underline{f}_x = \mathrm{col}(f_{x2}, f_{x3}, f_{x4}, f_{x5}, f_{x6}, f_{x7}).$$

In a like manner define a vector $\underline{f}_y$. For the counterclockwise external moments m_i applied between hinge i-1 and i, the vector

$$\underline{m} = \mathrm{col}(m_2, m_3, m_4, m_5, m_6, m_7)$$

is defined. The generalized force vector may be expressed in terms of these vectors as

$$\underline{q} = \underline{L}^t \underline{m} - Y^t \underline{f}_x + \underline{X}^t \underline{f}_y \ . \tag{4}$$

It should be noted that the matrices $\underline{J}$ and $\underline{K}$ are both symmetric. This condition results when any symmetric matrix is postmultiplied by a matrix and is premultiplied by its transpose. This symmetry property is highly desirable when dealing with matrix eigenvalue problems and should be preserved if possible when applying boundary conditions.

Application of Boundary Conditions

Up to this point the development has been general in nature. We now restrict further development to free vibration. With this restriction, the only external forces applied are those required to satisfy end constraints. If the model has both ends fixed, the end conditions and corresponding external forces are

$$\theta_1 = 0,\ m_2 \neq 0$$
$$v_7 = 0,\ f_{y7} \neq 0$$
$$u_7 = 0,\ f_{x7} \neq 0$$
$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 = 0,\ m_7 \neq 0 \ .$$

Equation (4) can therefore be reduced to

$$\underline{q} = \underline{Q}\,\underline{f} \qquad (5)$$

where

$$\underline{f} = \mathrm{col}(m_2,\ f_{y7},\ -f_{x7},\ m_7)$$

$$\underline{Q} = \begin{bmatrix} 1 & x_{61} & y_{61} & 1 \\ 0 & x_{62} & y_{62} & 1 \\ 0 & x_{63} & y_{63} & 1 \\ 0 & x_{64} & y_{64} & 1 \\ 0 & x_{65} & y_{65} & 1 \\ 0 & x_{66} & y_{66} & 1 \end{bmatrix} .$$

The columns of $\underline{Q}$ are the first and last columns of $\underline{L}^t$, and the last columns of $\underline{X}^t$ and $\underline{Y}^t$.

The end conditions may be represented by

$$\underline{Q^t\theta} = 0 \ . \qquad (6)$$

This equation is derived from the following set of equations.

$$\theta_1 = 0$$

$$u_7 = x_{61}\theta_1+x_{62}\theta_2+x_{63}\theta_3+x_{64}\theta_4+x_{65}\theta_5+x_{66}\theta_6 = 0$$

$$v_7 = y_{61}\theta_1+y_{62}\theta_2+y_{63}\theta_3+y_{64}\theta_4+y_{65}\theta_5+y_{66}\theta_6 = 0$$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 = 0$$

A process must be devised to incorporate the boundary conditions. This process will reduce the number of degrees of freedom by an amount equal to the number of constraint conditions, in this case four. The result will be a pair of homogeneous equations with two remaining θ_j's.

A normal Gaussian elimination technique for the solution of simultaneous equations is used to solve for f_{x7}, f_{y7}, m_2, and m_7. Through the use of the end conditions, the number of unknown θ_j is next reduced to two. The four non-homogeneous equations containing the forces and moments can be deleted, leaving only the two homogeneous equations. A similar elimination could be used for any other set of end conditions.

This process can be represented by a set of matrix operations. Equation (3) is written in partitioned form as

$$\begin{bmatrix} \underline{J}_u \\ \\ \underline{J}_\ell \end{bmatrix} \ddot{\underline{\theta}} + \begin{bmatrix} \underline{K}_u \\ \\ \underline{K}_\ell \end{bmatrix} \underline{\theta} = \begin{bmatrix} \underline{Q}_u \\ \\ \underline{Q}_\ell \end{bmatrix} \underline{f} \qquad (7)$$

where the subscript u represents the rows which will contain the non-homogeneous equations, and ℓ the remaining rows. This may require a row interchange to insure a non-singular $\underline{Q}_u$. In the present case $\underline{Q}_u$ is 4x4. Gaussian elimination is now performed to solve for the forces and moments. In this process the elements of $\underline{Q}_\ell$ will be reduced to zero. The lower portion of equation (7) may be rewritten

$$\underline{J}'_\ell \ddot{\underline{\theta}}_\ell + \underline{K}'_\ell \underline{\theta}_\ell = 0 \qquad (8)$$

where the prime denotes reduced matrices.

The end conditions, equation (6), may be rewritten

$$\begin{bmatrix} \underline{Q}_u^t & \underline{Q}_\ell^t \end{bmatrix} \begin{bmatrix} \underline{\theta}_u \\ \\ \underline{\theta}_\ell \end{bmatrix} = 0$$

and from this we get

$$\underline{Q}_u^t \underline{\theta}_u + \underline{Q}_\ell^t \underline{\theta}_\ell = 0$$

and therefore

$$\underline{\theta}_u = -(\underline{Q}_u^t)^{-1}\underline{Q}_\ell^t\underline{\theta}_\ell \quad .$$

Thus $\underline{\theta}_u$ may be eliminated from $\underline{\theta}$ by the transformation

$$\underline{\theta} = \begin{bmatrix} -(\underline{Q}_u^t)^{-1}\underline{Q}_\ell^t \\ \\ \underline{I} \end{bmatrix} \underline{\theta}_\ell = \underline{P}\underline{\theta}_\ell \tag{9}$$

where $\underline{I}$ is the identity matrix, in this case of order 2x2.

It is easily shown that

$$\underline{P}^t\underline{Q} = 0$$

and that in equation (8)

$$\underline{J}_\ell' = \underline{P}^t\underline{J}$$

and

$$\underline{K}_\ell' = \underline{P}^t\underline{K} \quad .$$

The $\underline{\theta}$ may then be replaced in equation (8) by $\underline{P}\underline{\theta}_\ell$. After differentiating equation (9) twice, the $\underline{\ddot{\theta}}$ of equation (8) may also be replaced, and the following expression obtained:

$$\underline{J}_\ell''\underline{\ddot{\theta}}_\ell + \underline{K}_\ell''\underline{\theta}_\ell = 0 \tag{10}$$

where

$$\underline{J}_\ell'' = \underline{P}^t\underline{JP}$$

$$\underline{K}_\ell'' = \underline{P}^t\underline{KP} \quad .$$

The boundary conditions can therefore be applied and a set of homogeneous equations obtained by this process. This transformation does not destroy the symmetry of the matrices.

Eigenvalue Solution of the Equations of Motion

Assuming that the vibration will be sinusoidal, equation (10) may be written

$$\underline{K}''_{\ell}\underline{\theta}_{\ell} = \omega^{2}\underline{J}''_{\ell}\underline{\theta}_{\ell} \tag{11}$$

where ω is the natural circular frequency. This equation is in a form suitable for eigenvalue solution.

The solution of equation (11) is now undertaken. This solution will involve eigenvalues and eigenvectors.[2] A symmetric matrix may be expressed in terms of its modal matrix and its spectral matrix where the spectral matrix is a diagonal matrix of eigenvalues and the columns of the modal matrix are eigenvectors. This property is used several times in the solution. A further important property of the modal matrix is that the eigenvectors are normalized with respect to the identity matrix. Thus the transpose of the modal matrix is also its inverse.

If $\underline{J}''_{\ell}$ is resolved into its (diagonal) spectral matrix $\underline{B}$ and modal matrix $\underline{D}$ according to the relation

$$\underline{J}''_{\ell} = \underline{DBD}^{t} \ ,$$

equation (11) becomes

$$\underline{K}''_{\ell}\underline{\theta}_{\ell} = \omega^{2}\underline{DBD}^{t}\underline{\theta}_{\ell} \ .$$

Since $\underline{DD}^{t} = \underline{I}$, the above equation may be rewritten

$$\underline{K}''_{\ell}\underline{DD}^{t}\underline{\theta}_{\ell} = \omega^{2}\underline{DBD}^{t}\underline{\theta}_{\ell} \ .$$

Since $\underline{B}$ is a diagonal matrix of positive elements, we define the principal square root as $\underline{B}^{\frac{1}{2}}$ where the elements of this matrix are the positive square roots of the elements of $\underline{B}$. Noting that $\underline{B}^{-\frac{1}{2}}\underline{B}^{\frac{1}{2}} = I$, the following form results,

$$\underline{K}''_{\ell}\underline{DB}^{-\frac{1}{2}}\underline{B}^{\frac{1}{2}}\underline{D}^{t}\underline{\theta}_{\ell} = \omega^{2}\underline{DB}^{\frac{1}{2}}B^{\frac{1}{2}}D^{t}\underline{\theta}_{\ell} \ .$$

Letting $\underline{z} = \underline{B}^{\frac{1}{2}}\underline{D}^{t}\underline{\theta}_{\ell}$,

$$\underline{K}''_{\ell}\underline{DB}^{-\frac{1}{2}}\underline{z} = \omega^{2}\underline{DB}^{\frac{1}{2}}\underline{z} \ .$$

Solving for $\omega^{2}\underline{z}$,

$$\underline{B}^{-\frac{1}{2}}\underline{D}^{t}\underline{K}''_{\ell}\underline{DB}^{-\frac{1}{2}}\underline{z} = \omega^{2}\underline{z} \ .$$

Letting $\underline{S} = \underline{B}^{-\frac{1}{2}}\underline{D}^{t}\underline{K}''_{\ell}\underline{DB}^{-\frac{1}{2}}$, the equation may be rewritten

$$\underline{Sz} = \omega^{2}\underline{z} \ .$$

By diagonalization of $\underline{S}$, the values of ω^{2} and $\underline{z}$ may be obtained. From these values, the natural frequencies are obtained directly. The values of the rotations of the nodes are obtained from the equation $\underline{\theta}_{\ell} = \underline{DB}^{\frac{1}{2}}\underline{z}$. The rotations at all the hinges of the structure can be recovered by use of equation (9). The translation displacements of the hinges are easily obtained with the use of equations (1).

DEVELOPMENT OF FINITE-ELEMENT-MODEL

General

A finite-element-model is developed to describe the in-plane motion of a frame with straight members of uniform cross section. The model neglects axial and shear deformation and can include rotatory inertia if desired.

The use of a finite element approach requires the division of the structure into elements, and the determination of element mass and stiffness matrices. The determination of the elemental mass and stiffness matrices will be taken up first.

Consider a general element of length ℓ, in a local coordinate system. In this system, r is the axial coordinate, $\underset{\sim}{w}$ is the transverse deflection, and ϕ is the slope. The element has no flexibility in the r direction. If we assume that w may be approximated by a third degree polynomial in r, then

$$w = c_o + c_1 r + c_2 r^2 + c_3 r^3$$

and

$$\phi = \frac{dw}{dr} = c_1 + 2c_2 r + 3c_3 x^2$$

where c_o, c_1, c_2, c_3 are unknown coefficients. These equations may be expressed as

$$w = \underline{p}^t \underline{c}$$

$$\phi = \underline{h}^t \underline{c}$$

where

$$\underline{p}^t = \text{row}(1,\ r,\ r^2,\ r^3)$$

$$\underline{h}^t = \text{row}(0,\ 1,\ 2r,\ 3r^2)$$

$$\underline{c} = \text{col}(c_o,\ c_1,\ c_2,\ c_3)\ .$$

If w and ϕ are evaluated at the ends of the element, the results may be expressed as

$$\underline{d} = \underline{C}^{-1}\underline{c}$$

where

$$\underline{d} = \mathrm{col}(w_0,\ \phi_0,\ w_\ell,\ \phi_\ell)$$

$$\underline{C}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix}$$

and the subscripts denote evaluation at r=0, and r=ℓ. By inverting $\underline{C}^{-1}$ we may solve for $\underline{c}$,

$$\underline{c} = \underline{Cd} \qquad (12)$$

where

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{\ell^2} & \frac{-2}{\ell} & \frac{3}{\ell^2} & \frac{-1}{\ell} \\ \frac{2}{\ell^3} & \frac{1}{\ell^2} & \frac{-2}{\ell^3} & \frac{1}{\ell^2} \end{bmatrix} .$$

Bending Energy

The bending energy, U, of the element is given by

$$U = \frac{1}{2}\int_0^\ell EI\left(\frac{d^2w}{dr^2}\right)^2 dr \qquad (13)$$

where EI is the flexural rigidity for in-plane bending.

Now $\dfrac{d^2w}{dr^2} = 2c_2 + 6c_3r = \underline{g}^t\underline{c}$

where $\underline{g}^t = \mathrm{row}(0,\ 0,\ 2,\ 6r)$.

Since EI is a scalar constant, the equation may be rewritten

$$U = \frac{1}{2}EI\int_0^{\ell}\underline{c}^t\underline{gg}^t\underline{c}dr .$$

Since $\underline{c}$ and $\underline{c}^t$ are constants, they may be removed from the integral. Substituting equation (12) for $\underline{c}$ and $\underline{c}^t$ we may write

$$U = \frac{1}{2}\underline{d}^t\underline{K}'\underline{d} \tag{14}$$

where

$$\underline{K}' = \text{bending stiffness} = EI\underline{C}^t\left(\int_0^{\ell}\underline{gg}^t dr\right)\underline{C} . \tag{15}$$

It can be shown by carrying out the required integration and matrix multiplications that

$$\underline{K}' = \left(EI/\ell^3\right)\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell^2 & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \tag{16}$$

Kinetic Energy

The kinetic energy, T, of the element is given by

$$T = \frac{1}{2}\mu\int_0^{\ell}\dot{w}^2 dr + \frac{1}{2}\frac{\mu I}{A}\int_0^{\ell}\dot{\phi}^2 dr \tag{17}$$

where

μ = mass per unit length

A = cross sectional area

I = second moment of area A about an axis through the centerline of the element and perpendicular to both the w and r axes.

The first term is the translation energy, T_t, and the second term is the rotational energy, T_r. Thus

$$T_t = \frac{1}{2}\mu\int_0^{\ell}\dot{w}^2 dr$$

and

$$T_r = \frac{1}{2}\,\frac{\mu I}{A}\int_0^{\ell}\dot{\phi}^2 dr \;.$$

Now

$$\begin{aligned}\dot{w} &= \dot{c}_0 + \dot{c}_1 r + \dot{c}_2 r^2 + \dot{c}_3 r^3 \\ &= \underline{p}^t\underline{\dot{c}} = \underline{p}^t\underline{C}\underline{\dot{d}} \;.\end{aligned}$$

Again noting that $\underline{C}$ and $\underline{C}^t$ do not depend on r, T_t may be rewritten

$$T_t = \tfrac{1}{2}\underline{\dot{d}}^t\underline{M}'_t\underline{\dot{d}} \tag{18}$$

where

$$\underline{M}'_t = \mu\underline{C}^t\left(\int_0^{\ell}\underline{p}\underline{p}^t dr\right)\underline{C} \;. \tag{19}$$

Using

$$\begin{aligned}\dot{\phi} &= \dot{c}_1 + 2\dot{c}_2 r + 3\dot{c}_3 r^2 \\ &= \underline{h}^t\underline{\dot{c}} = \underline{h}^t\underline{C}\underline{\dot{d}} \;,\end{aligned}$$

by a similar process

$$T_r = \tfrac{1}{2}\underline{\dot{d}}^t\underline{M}'_r\underline{\dot{d}} \tag{20}$$

where

$$\underline{M}'_r = \frac{\mu I}{A}\,\underline{C}^t\left(\int_0^{\ell}\underline{h}\underline{h}^t dr\right)\underline{C} \;. \tag{21}$$

After evaluation

$$\underline{M}'_t = (\mu\ell/420)\begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \tag{22}$$

and

$$\underline{M}'_r = (\mu I/30A\ell)\begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ 3\ell & 4\ell^2 & -3\ell & -\ell^2 \\ -36 & -3\ell & 36 & -3\ell \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix} \;. \tag{23}$$

If equations (18) and (20) are recombined, then

$$T = \frac{1}{2}\underline{\dot{d}}^t \underline{M}'\underline{\dot{d}} \tag{24}$$

where

$$\underline{M}' = \underline{M}'_t + \underline{M}'_r \tag{25}$$

and $\underline{M}'$ is called the consistent mass matrix for bending.[3]

Axial Displacement

Up to this point there has been no mention of a displacement in the axial direction. Since the element has no axial flexibility, any displacement in the axial direction will be displacement of the element as a rigid body. This displacement will be the same at either end of the element. The axial displacement is represented by u, where $u=u_o=u_\ell$. In fact, as will be shown later, the axial displacement of an entire straight member made up of several elements may be represented by a single symbol.

The axial displacement should be included in the bending and kinetic energy equations. The u has no effect on the bending energy or rotational energy. It does contribute to the kinetic energy of translation. This contribution is $\frac{1}{2}\mu\ell\dot{u}^2$. Equations (14) and (24) may be rewritten

$$U = \frac{1}{2}\,\underline{v}^t\underline{Kv} \tag{26}$$

$$T = \frac{1}{2}\,\underline{\dot{v}}^t\underline{M\dot{v}} \tag{27}$$

where

$$\underline{v} = \mathrm{col}(u,\ w_0,\ \phi_0,\ w_\ell,\ \phi_\ell)$$

$$\underline{K} = \left[\begin{array}{c:cccc} 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & & & & \\ 0 & & \underline{K}' & & \\ 0 & & & & \\ 0 & & & & \end{array}\right] \tag{29}$$

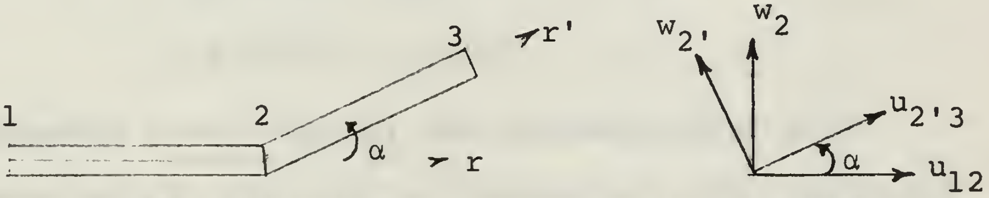
$$\underline{M} = \underline{M}_t + \underline{M}_r \tag{30}$$

$$\underline{M}_t = \left[\begin{array}{c:cccc} \mu\ell & 0 & 0 & 0 & 0 \\ \hdashline 0 & & & & \\ 0 & & \underline{M}'_t & & \\ 0 & & & & \\ 0 & & & & \end{array}\right]$$

$$\underline{M}_r = \left[\begin{array}{c:cccc} 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & & & & \\ 0 & & & & \\ 0 & & \underline{M}'_r & & \\ 0 & & & & \end{array}\right] .$$

Corner Transformation

The combination of the element kinetic and bending energy to form the total bending energy and kinetic energy of the structure is not difficult, except at a corner. Figure 2a shows two elements, one to the left and one to the right of the corner. A subscripting notation has been added. A single subscript is used to denote the end or node at which the quantity is evaluated. A double subscript is used to denote quantities which are the same at each end of an element, and for vectors and matrices which pertain to the element. Thus $\underline{v}_{12} = \text{col}(u_{12}, w_1, \phi_1, w_2, \phi_2)$, and so on.



(a) Elements Adjacent to Corner

(b) Coordinates at Node 2

Figure 2

The displacements at node 2 are with reference to two different local coordinate systems, and cannot be directly combined (Figure 2b). The prime is used to denote the displacements at node 2 which are in the local coordinate system of the element to the right of the corner. There are only two independent displacements at node 2. It is thus necessary to express the four displacements at this node in terms of only two displacements. The axial displacements of the two elements are chosen. This choice is desirable since the axial displacement of all parts of a straight member is the same. Further, when end conditions are applied, the axial displacements at the ends are some of the quantities which are known.

Letting $\beta = \cos\alpha$ and $\eta = \sin\alpha$, from figure 2b we can write

$$w_{2'} = \beta w_2 - \eta u_{12}$$

$$u_{2'3} = \eta w_2 + \beta u_{12} \ .$$

Using the above equations, w_2 and $w_{2'}$ may be expressed in terms of u_{12} and $u_{2'3}$ as:

$$w_{2'} = -(1/\eta)u_{12} + (\beta/\eta)u_{2'3} \qquad (31)$$

$$w_2 = -(\beta/\eta)u_{12} + (1/\eta)u_{2'3} \qquad (32)$$

This transformation may be put into a convenient matrix notation. For the element to the left of the corner, this transformation is written

$$\begin{aligned} \underline{v}_{12} &= \underline{R}\underline{v}^*_{12} \\ \underline{v}^t_{12} &= (\underline{v}^*_{12})^t \underline{R}^t \end{aligned} \qquad (33)$$

where

$$\underline{v}^*_{12} = \mathrm{col}(u_{12},\ w_1,\ \phi_1,\ u_{2'3},\ \phi_2)$$

$$\underline{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{\beta}{\eta} & 0 & 0 & \frac{1}{\eta} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

The kinetic and potential energy of the element may therefore be written

$$\begin{aligned} T_{12} &= \frac{1}{2}(\dot{\underline{v}}^*_{12})^t \underline{R}^t \underline{M}_{12} \underline{R} \dot{\underline{v}}^*_{12} \\ U_{12} &= \frac{1}{2}(\underline{v}^*_{12})^t \underline{R}^t \underline{K}_{12} \underline{R} \underline{v}^*_{12} \quad . \end{aligned} \tag{34}$$

For the element to the right of the corner, the transformation is

$$\begin{aligned} \underline{v}_{2'3} &= \underline{R}' \underline{v}^*_{2'3} \\ \underline{v}_{2'3} &= (\underline{v}^*_{2'3})^t (\underline{R}')^t \end{aligned} \tag{35}$$

where

$$\underline{v}^*_{2'3} = \mathrm{col}(u_{12},\ u_{2'3},\ \phi_2,\ w_3,\ \phi_3)$$

$$\underline{R}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{\eta} & \frac{\beta}{\eta} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

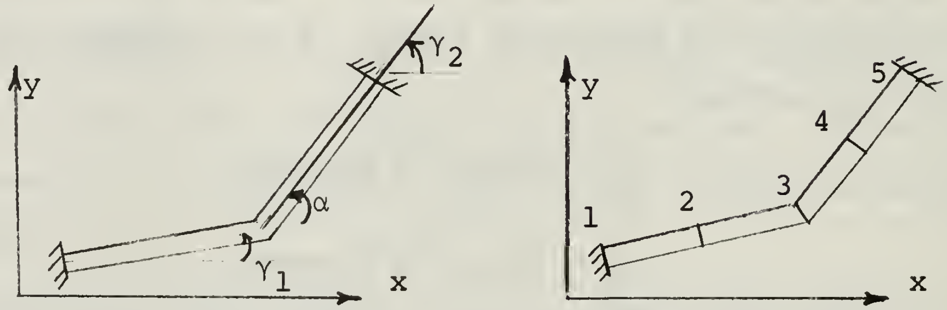
The kinetic and bending energy for the element to the right of the corner are written

$$T_{2'3} = \frac{1}{2}(\dot{\underline{v}}^*_{2'3})^t(\underline{R}')^t\underline{M}_{2'3}\underline{R}'\dot{\underline{v}}^*_{2'3}$$

$$U_{2'3} = \frac{1}{2}(\underline{v}^*_{2'3})^t(\underline{R}')^t\underline{K}_{2'3}\underline{R}'\underline{v}^*_{2'3} \; . \qquad (36)$$

The kinetic and bending energies in equations (34) and (36) can now be combined.

Example of Method



(a) Two Member Frame

(b) Frame Divided Into Elements

Figure 3

To illustrate the method, a frame consisting of two straight members intersecting at an acute angle, α, is used (Figure 3a). Both ends of the structure are fixed.

Each member is divided into 2 elements (Figure 3b). Each of the two members need not have been divided into the same number of elements. The nodes at the ends and between elements are numbered for identification. The elemental mass and stiffness matrices are determined and the following set of equations written:

$$T_{12} = \frac{1}{2}\dot{\underline{v}}^t_{12}\underline{M}_{12}\dot{\underline{v}}_{12} \; , \; U_{12} = \frac{1}{2}\underline{v}^t_{12}\underline{K}_{12}\underline{v}_{12} \qquad (37)$$

$$T_{23} = \frac{1}{2}\dot{\underline{v}}^t_{23}\underline{M}_{23}\dot{\underline{v}}_{23} \ , \ U_{23} = \frac{1}{2}\underline{v}^t_{23}\underline{K}_{23}\underline{v}_{23} \tag{38}$$

$$T_{34} = \frac{1}{2}\dot{\underline{v}}^t_{34}\underline{M}_{34}\dot{\underline{v}}_{34} \ , \ U_{34} = \frac{1}{2}\underline{v}^t_{34}\underline{K}_{34}\underline{v}_{34} \tag{39}$$

$$T_{45} = \frac{1}{2}\dot{\underline{v}}^t_{45}\underline{M}_{45}\dot{\underline{v}}_{45} \ , \ U_{45} = \frac{1}{2}\underline{v}^t_{45}\underline{K}_{45}\underline{v}_{45} \ . \tag{40}$$

The corner transformation must be performed at node 3. Equations (38) and (39) become

$$T_{23} = \frac{1}{2}(\dot{\underline{v}}^*_{23})^t\underline{R}^t\underline{M}_{23}\underline{R}\dot{\underline{v}}^*_{23} \ , \ U_{23} = \frac{1}{2}(\underline{v}^*_{23})^t\underline{R}^t\underline{K}_{23}\underline{R}\underline{v}^*_{23} \tag{41}$$

$$\begin{gathered} T_{3'4} = \frac{1}{2}(\dot{\underline{v}}^*_{3'4})^t(\underline{R}')^t\underline{M}_{3'4}\underline{R}'\dot{\underline{v}}^*_{3'4} \ , \\ U_{3'4} = \frac{1}{2}(\underline{v}^*_{3'4})^t(\underline{R}')^t\underline{K}_{3'4}\underline{R}'\underline{v}^*_{3'4} \ . \end{gathered} \tag{42}$$

The elemental kinetic and bending energy equations are combined and are written

$$\begin{gathered} T_a = \frac{1}{2}\dot{\underline{v}}^t_a\underline{M}_a\dot{\underline{v}}_a \\ U_a = \frac{1}{2}\underline{v}^t_a\underline{K}_a\underline{v}_a \end{gathered} \tag{43}$$

where

$\underline{v}_a = \mathrm{col}(u_{13}, w_1, \phi_1, w_2, \phi_2, u_{35}, \phi_3, w_4, \phi_4, w_5, \phi_5)$

$\underline{M}_a$ = assembled mass matrix

$\underline{K}_a$ = assembled stiffness matrix

and the subscript a denotes assembled quantities. The assembled matrices are formed by placing the elemental matrices in their correct positions in the larger framework of the assembled matrix, and summing all the overlapping terms.[4]

The Equations of Motion

Using the Lagrangian equation of motion,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\underline{v}}}\right) - \frac{\partial T}{\partial \underline{v}} + \frac{\partial U}{\partial \underline{v}} = \underline{q}$$

equation (43) may be written

$$\underline{M}_a \ddot{\underline{v}}_a + \underline{K}_a \underline{v}_a = \underline{q} \qquad (44)$$

where $\underline{q}$ is the generalized force vector. The elements of $\underline{q}$ are the generalized forces and moments corresponding to the elements of the displacement vector, $\underline{v}_a$. This equation is general in that no restriction has been placed on the generalized force vector.

End Conditions

We now restrict further development to free vibration only. Since in Figure 3 both ends are fixed and there is no external loading, the only non-zero forces in the vector $\underline{q}$ are the forces required to satisfy end constraints.

For both ends fixed, the following is known about the structure:

$$u_{13} = w_1 = \phi_1 = 0$$

$$u_{35} = w_5 = \phi_5 = 0 \ .$$

The order of the matrices in equation (44) is now reduced by an amount equal to the number of constraint conditions, in this case six. The rows and columns corresponding to these conditions are deleted and the equation rewritten

$$\underline{M}''_a \ddot{\underline{v}}''_a + \underline{K}''_a \underline{v}''_a = 0 \qquad (45)$$

where

$$\underline{v}_a'' = \mathrm{col}(w_2,\ \phi_2,\ \phi_3,\ w_4,\ \phi_4)$$

$$\underline{K}_a'' = \text{a 5x5 reduced matrix}$$

$$\underline{M}_a'' = \text{a 5x5 reduced matrix.}$$

Eigenvalue solution

Assuming the vibrational motion is sinusoidal, the equation becomes

$$\underline{K}_a''\underline{v}_a'' = \omega^2\underline{M}_a''\underline{v}_a'' \qquad (46)$$

where ω is the natural circular frequency. This equation is now in a form suitable for eigenvalue solution. The solution is carried out in the same manner as described on pages 24 to 25 of the lumped parameter development. This solution will yield the first five natural frequencies (since matrices are 5x5), the transverse displacements at the 2nd and 4th node, and the slopes at nodes 2, 3, 4. The displacements are known at the ends from the end conditions, and through use of equation (31) or (32) the displacement at node 3 may be obtained. These displacements are all referred to local coordinate systems. They are transformed to a Cartesian (x,y,z) system through the use of the following equations:

$$u_j = -w_j \sin \gamma_i + u_j \cos \gamma_i$$

$$v_j = w_j \cos \gamma_i + u_j \sin \gamma_i$$

where

γ_i = angle between the x axis and axial direction of member i

u_j = displacement in x direction

v_j = displacement in y direction

j = node number, $j=1,2,3,4,5$

i = member number, $i=1,2$.

PRINCIPAL MODE COMPARISON

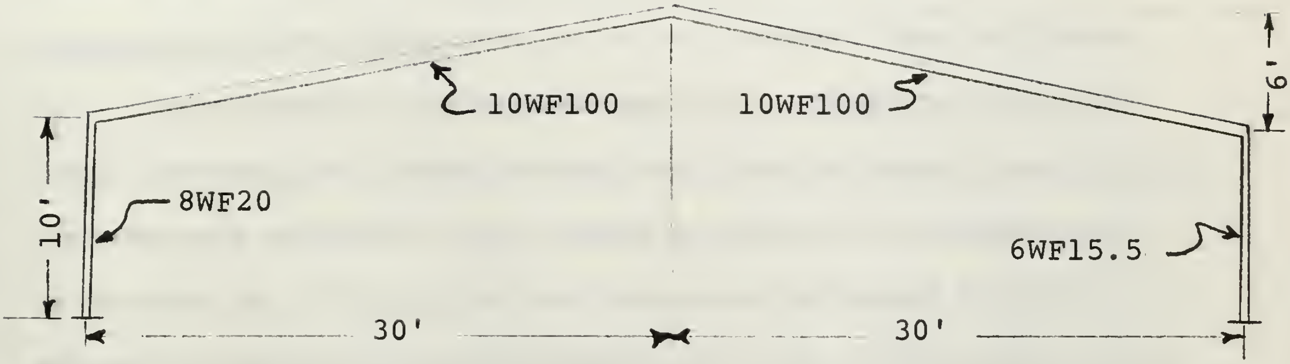


Figure 4

Gable Bent Used in Analyses

Figure 4 shows the structure used for comparison of the models. The members are all steel wide-flanged I-beams of the type shown on the figure. Data for these I-beams is taken from reference 5.

Transfer Matrix Method.

The transfer matrix method is widely used for the determination of the principal free vibrational modes of framed structures and will be used as the standard for comparing the finite-element and lumped-parameter-models. The development and use of the transfer matrix method are given detailed treatment in reference 1. A computer program to analyze similar structures was developed in 1964 by Fink in his master's thesis at the Naval Postgraduate School.[6] The "exact values" used to compare the models are generated through the use of a transfer matrix computer program developed by the author. This program is included as Appendix D.

The first five undamped natural frequencies and corresponding mode shapes are calculated using this program for four end conditions: clamped-clamped, clamped-simply supported, pinned-pinned, and pinned-simply supported. The frequencies are given in Table I and the mode shapes are listed in Appendix A, pages 50, 51, 52, 53. In Figure 5, the mode shapes for the clamped-clamped end condition are plotted. The displacements are greatly exaggerated so that the mode shape can be more readily visualized. Only the first five frequencies are tabulated, since after this the computer round-off error begins to affect the mode shape. The frequencies have become large enough that, even though the convergence routine has estimated them to nine decimal places, the frequency determinant is not zero. Thus when the state vector of the left end is passed through the transfer matrices, the accumulation of round-off error is evident at the right end. The beginnings of this error can be seen in the computer output for the clamped-clamped case (page 50). In the fifth mode, the x displacement is 0.0014 at the right end, but should be zero since the end is fixed. This susceptibility of the transfer matrix method to round-off error is a major reason why other computerized models are sought.

The structure was turned around and the frequencies and mode shapes calculated for the pinned-pinned and clamped-clamped end conditions. The frequencies are unchanged, but the fifth mode shape differs slightly in the fourth decimal place. This difference is due to the accumulated round-off error. The outputs for these runs are listed in Appendix D on page 112.

Lumped-Parameter-Model

An analysis of the same gable bent using the lumped-parameter-model is next carried out. Since both ends clamped is the most severely constrained condition, this end condition will cause the structure to have the fewest degrees of freedom. Using this end condition, it is expected that calculation of the natural frequencies and displacements will require the greatest number of elements. Therefore, by successive trials, sufficient subdividing is carried out such that the error in the primary frequency is less than 1% and the error in the 5th frequency is less than 3%. To meet these criteria requires 38 subdivisions. Using the same number of subdivisions, the rest of the end conditions are run and the results given in Table I. The pinned-pinned end condition is seen to lead to a slightly greater per cent difference than the clamped-clamped condition.

A comparison of the mode shapes for this model with those of the standard is made by comparing the outputs on pages 50, 51, 52, and 53 in Appendix A. The mode shapes agree to two decimal places for the first four modes, and to only one decimal place in the fifth mode.

The error due to the lumping of the parameters decreases slowly as the number of subdivisions increases. By comparing

Plot of Mode Shapes for a Gable Bent With Both Ends Fixed.

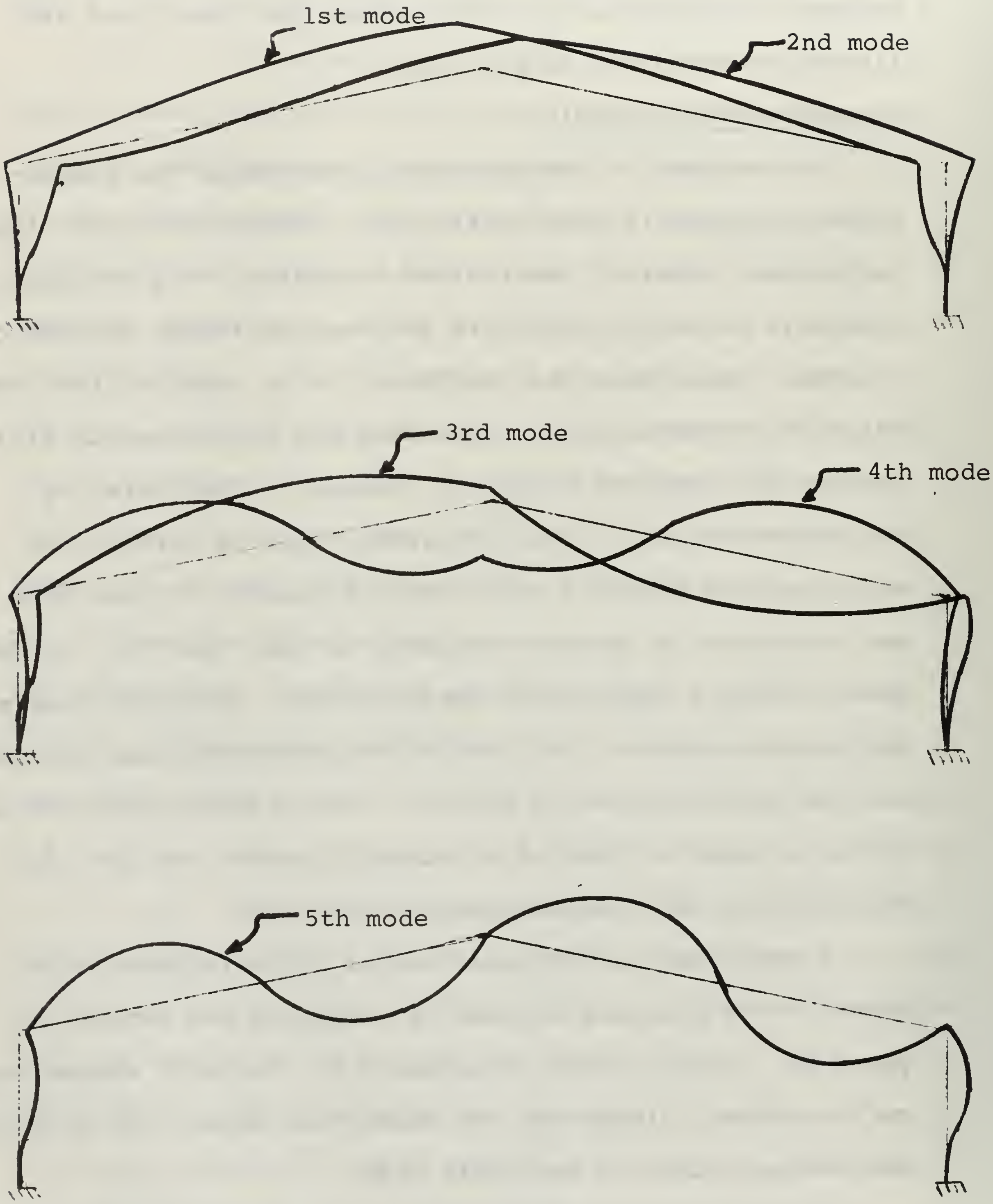


Figure 5

page 76 to the values in Table I, the small improvement in modal frequencies resulting from increasing the number of subdivisions can be seen. There is a similar improvement in mode shapes. If an accurate estimate of frequency and displacement were desired, a very high degree of discretization would be needed. The computer time necessary to solve for these values would be large. This is due to the eigenvalue routine where the time for solution varies as the cube of the number of subdivisions.

The frame was turned around and values obtained for the pinned-pinned and clamped-clamped end conditions. These frequencies and displacements (page 75) are identical to those calculated in Table I. This duplication of results adds a measure of confidence to the integrity of the solution using this model.

Finite Element Model

The frame is again analyzed, this time using the finite-element-model. The analysis is started using both ends clamped. To meet the criteria of less than 1% difference in the primary frequency and less than 3% difference in the 5th frequency requires 12 elements. The frequencies for all end conditions using 12 elements are listed in Table I.

The mode shapes obtained by this method are in excellent agreement with those of the standard. A comparison of the mode shapes from this method to those of the standard (Appendix A, pages 50, 51, 52, and 53) shows agreement to three decimal places.

TABLE I

Summary of Modal Frequencies

Clamped-Clamped End Condition

Mode Number	Transfer Matrix	Lumped-Parameter Sections:6-13-13-6		Finite-Element Sections:2-4-4-2	
	Frequency	Frequency	% dif.	Frequency	% dif.
1	25.754	25.921	+0.65	25.752	-0.008
2	34.680	34.929	+0.72	34.679	-0.003
3	79.630	79.366	-0.33	79.647	+0.020
4	150.284	148.262	-1.34	150.444	+0.110
5	279.474	271.688	-2.78	280.597	+0.410

Clamped-Simple End Condition

Mode Number	Transfer Matrix	Lumped-Parameter Sections:6-13-13-6		Finite-Element Sections:2-4-4-2	
	Frequency	Frequency	% dif.	Frequency	% dif.
1	14.765	14.769	+0.02	14.764	-0.008
2	34.238	34.482	+0.71	34.236	-0.006
3	75.052	74.760	-0.39	75.065	+0.018
4	123.234	123.869	+0.51	123.283	+0.040
5	152.589	150.914	-1.11	152.752	+0.100

Pinned-Pinned End Condition

Mode Number	Transfer Matrix	Lumped-Parameter Sections:6-13-13-6		Finite-Element Sections:2-4-4-2	
	Frequency	Frequency	% dif.	Frequency	% dif.
1	15.257	15.285	+0.184	15.256	-0.008
2	25.365	25.365	0.00	25.364	-0.006
3	75.652	75.162	-0.65	75.667	+0.020
4	149.662	147.574	-1.39	149.820	+0.110
5	277.069	269.188	-2.84	278.160	+0.390

Pinned-Simple End Condition

Mode Number	Transfer Matrix	Lumped-Parameter Sections:6-13-13-6		Finite-Element Sections:2-4-4-2	
	Frequency	Frequency	% dif.	Frequency	% dif.
1	9.742	9.747	+0.051	9.741	-0.008
2	24.419	24.419	0.00	24.417	-0.006
3	72.489	72.041	-0.62	72.502	+0.018
4	122.907	123.511	+0.49	122.956	+0.040
5	152.252	150.555	-1.11	152.412	+0.110

As the number of elements used increases, the differences between the calculated values and the standard show a marked decrease. The increasing of the subdivisions from 8 to 12 decreases the per cent difference by a factor of 10. The results for the 8 element structure are on page 96 of Appendix C. This marked decrease in the per cent difference illustrates the fact that this model will give good results with little subdividing.

It is known that the finite-element-method is equivalent to the Rayleigh-Ritz procedure for finding natural frequencies.[3] Accordingly the estimate for the frequencies should represent an upper bound. The values in Table I do not agree with this. In all cases, the first and second frequencies from the finite-element-model are slightly lower than those of the transfer matrix method. An attempt to find a reason for this discrepancy in the programming of the models was unsuccessful.

The pinned-pinned and clamped-clamped end conditions were rerun with the structure reversed. The frequencies and mode shapes were identical to those previously calculated (Appendix C, page 95).

Using the finite-element-model with 12 elements, frequencies and mode shapes were calculated with rotatory inertia included. The results are given in Table II. These results show that rotatory inertia has little effect on the frequencies. The maximum difference between frequencies with and without rotatory inertia is 0.3%.

TABLE II

Effect of Rotatory Inertia on Frequency

End Condition		Frequency (rad/sec)				
		1	2	3	4	5
Clamped Clamped	W/O With % dif.	25.752 25.748 0.015	34.679 34.675 0.012	79.647 79.585 0.078	150.444 150.207 0.158	280.597 279.725 0.312
Clamped Simple	W/O With % dif.	14.764 14.762 0.013	34.236 34.232 0.011	75.065 75.008 0.076	123.283 123.147 0.110	152.752 152.516 0.155
Pinned Pinned	W/O With % dif.	15.256 15.255 0.006	25.364 25.359 0.020	75.667 75.607 0.078	149.820 149.583 0.158	278.160 277.296 0.310
Pinned Simple	W/O With % dif.	9.741 9.740 0.001	24.417 24.413 0.016	72.502 72.415 0.079	122.956 122.819 0.112	152.412 152.177 0.159

Comparison of Models

Both the lumped-parameter and finite-element-models lend themselves favorably to computer analysis. The parameters needed for analysis are easily derived and fit conveniently into a matrix form which is easily programmed. Each model has one feature which may cause some difficulty. These are the application of end conditions in the lumped-parameter-model and the corner transformation in the finite-element-model. Once understood, however, both of these can be programmed.

The inertial and stiffness matrices resulting from these models are quite different. The matrices from the finite-element-model are sparse. The matrices from the lumped-parameter-model do not have this property. The eigenvalue solution will therefore be easier and less time consuming for the finite-element-model since the matrices are already partially diagonalized.

The order of the resulting matrices is also different. The order of the finite-element-matrices is 2N+3 where N is the number of elements. The order of the lumped-parameter-matrices is N+1. The finite-element-model using 12 elements or 27 degrees of freedom yields much better solutions than the lumped-parameter-model with 38 segments or 39 degrees of freedom. This significant difference in the order of the matrices will save computation time also.

Conclusions

1. Both models are conveniently treated by digital computer methods and could be used for the analysis of frames.
2. Both models can include rotatory inertia.
3. Both models provide a basis for finding dynamic response of the structure to arbitrary time-varying external loads.
4. A comparison of the results of an analysis of a gable bent using the models shows that the finite-element-model gives much better results. It requires 12 fewer degrees of freedom and its average difference from the standard is one-tenth that of the lumped-parameter-model.
5. The transfer matrix method still gives good results for frequencies higher than the fifth, but not mode shapes. The finite-element-model gives good agreement at these higher frequencies, and its mode shapes should be accurate since the particular round-off error that affects the transfer matrix method does not pertain to the finite-element-model.

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APPENDIX A

PRINCIPAL MODE COMPARISON DATA

```
TPANSFER MATPIX METHOO
       X COORD(IN)          Y COORO(IN)   MOOULUS OF ELASTICITY(PSI)   SPECIFIC WT(LB/CU IN)   AREA(SO IN)        MOMENT(IN 4)
          0.0                  0.0
                                             30000000.00000                0.28300              5.88000            82.50000
          0.0                 60.00000
                                             30000000.00000                0.28300              5.88000            82.50000
          0.0                120.00000
                                             30000000.00000                0.28300             29.40000           625.00000
        180.00000            156.00000
                                             30000000.00000                0.28300             29.40000           625.00000
        360.00000            192.00000
                                             30000000.00000                0.28300             29.40000           625.00000
        540.00000            156.00000
                                             30000000.00000                0.28300             29.40000           625.00000
        720.00000            120.00000
                                             30000000.00000                0.28300              4.62000            30.30000
        720.00000             60.00000
                                             30000000.00000                0.28300              4.62000            30.30000
        720.00000              0.0
CLAMPEO CLAMPEO ENO CONOITION
ROTATOPY INEPTIA IS NOT INCLUOEO

NATURAL FREQUENCIES(RAO/SEC)
              1             2              3              4              5
  1         25.754        34.680         79.630        150.284        279.474

NORMALIZEO X OISPLACEMENTS
           1         2         3         4         5         6         7         8         9
  1     0.0     -0.0365   -0.1476   -0.2313   -0.2578   -0.3016   -0.3680   -0.2051   -0.0000
  2     0.0      0.1980    0.4195    0.3988    0.3440    0.3467    0.2684    0.0932    0.0000
  3     0.0      0.1604    0.1948    0.0614    0.1813    0.0422    0.1678    0.1616   -0.0000
  4     0.0      0.0587   -0.1077   -0.2129    0.0127    0.2284    0.1330   -0.0551   -0.0000
  5     0.0      0.6216    0.0226   -0.0149    0.0352    0.0435    0.0479    0.7535   -0.0014

NORMALIZED Y OISPLACEMENTS
           1         2         3         4         5         6         7         8         9
  1     0.0      0.0000    0.0000    0.4183    0.5509    0.3317   -0.0000   -0.0000   -0.0000
  2     0.0     -0.0000   -0.0000    0.1038    0.3777    0.3912    0.0000    0.0000    0.0000
  3     0.0     -0.0000   -0.0000    0.6670    0.0674   -0.6279   -0.0000   -0.0000   -0.0000
  4     0.0     -0.0000    0.0000    0.5262   -0.6017    0.4770   -0.0000   -0.0000   -0.0000
  5     0.0     -0.0000   -0.0000    0.1875   -0.0633   -0.0219   -0.0000   -0.0000   -0.0000
                                  ________________________________________

LUMPEO PAPAMETER METHOO
       X COORD(IN)          Y COORD(IN)         NUMBEP OF SECTIONS   SPECIFIC WT(LB/CU IN)   AREA(SO IN)        MOMENT(IN 4)
          0.0                  0.0
                                                     6.00000               0.28300              5.88000            82.50000
          0.0                120.00000
                                                    13.00000               0.28300             29.40000           625.00000
        360.00000            192.00000
                                                    13.00000               0.28300             29.40000           625.00000
        720.00000            120.00000
                                                     6.00000               0.28300              4.62000            30.30000
        720.00000              0.0
CLAMPED CLAMPEO ENO CONOITION
FOTATOPY INERTIA IS NOT INCLUOEO

NATUPAL FPEQUENCIES(RAO/SEC)
              1             2              3              4              5
  1         25.921        34.929         79.366        148.262        271.688

NOPMALIZEO X OISPLACEMENTS
           1         2         3         4         5         6         7         8         9
  1     0.0      0.0352    0.1452    0.2294    0.2558    0.2999    0.3664    0.2042   -0.0000
  2     0.0      0.1969    0.4192    0.4000    0.3444    0.3483    0.2696    0.0934    0.0000
  3     0.0     -0.1613   -0.1984   -0.0653   -0.1845   -0.0455   -0.1706   -0.1623    0.0000
  4     0.0      0.0600   -0.1054   -0.2119    0.0134    0.2284    0.1323   -0.0560   -0.0000
  5     0.0      0.6208    0.0247   -0.0160    0.0379    0.0464    0.0511    0.7492    0.0000

NORMALIZEO Y OISPLACEMENTS
           1         2         3         4         5         6         7         8         9
  1     0.0      0.0       0.0      -0.4209   -0.5531   -0.3326    0.0000   -0.0000   -0.0000
  2     0.0      0.0       0.0       0.0964    0.3741    0.3936   -0.0000   -0.0000   -0.0000
  3     0.0      0.0       0.0      -0.6654   -0.0693    0.6255    0.0000    0.0000    0.0000
  4     0.0      0.0       0.0       0.5324   -0.5943    0.4304   -0.0000    0.0000    0.0000
  5     0.0      0.0       0.0       0.2034   -0.0658   -0.0232    0.0000    0.0000    0.0000
                                  ________________________________________

FINITE ELEMENT METHOO
       X COORD(IN)          Y COCRO(IN)         NUMBEP OF SECTIONS   SPECIFIC WT(LB/CU IN)   APEA(SO IN)        MOMENT(IN 4)
          0.0                  0.0
                                                     2.00000               0.28300              5.88000            82.50000
          0.0                120.00000
                                                     4.00000               0.28300             29.40000           625.00000
        360.00000            192.00000
                                                     4.00000               0.28300             29.40000           625.00000
        720.00000            120.00000
                                                     2.00000               0.28300              4.62000            30.30000
        720.00000              0.0
CLAMPED CLAMPEO ENO CONOITION
ROTATORY INERTIA IS NOT INCLUDEO

NATURAL FREQUENCIES(RAO/SEC)
              1             2              3              4              5
  1         25.752        34.679         79.647        150.444        280.597

NORMALIZED X OISPLACEMENTS
           1         2         3         4         5         6         7         8         9
  1     0.0     -0.0365   -0.1476   -0.2313   -0.2578   -0.3016   -0.3680   -0.2051    0.0
  2     0.0      0.1980    0.4196    0.3988    0.3440    0.3468    0.2685    0.0932    0.0
  3     0.0     -0.1604   -0.1946   -0.0612   -0.1812   -0.0421   -0.1677   -0.1615    0.0
  4     0.0     -0.0587    0.1077    0.2130   -0.0126   -0.2284   -0.1330    0.0551    0.0
  5     0.0     -0.6213   -0.0222    0.0153   -0.0348   -0.0431   -0.0475   -0.7539    0.0

NORMALIZEO Y OISPLACEMENTS
           1         2         3         4         5         6         7         8         9
  1     0.0      0.0       0.0       0.4183    0.5509    0.3317   -0.0000    0.0       0.0
  2     0.0      0.0       0.0       0.1036    0.3775    0.3912    0.0000    0.0       0.0
  3     0.0      0.0       0.0      -0.6671   -0.0674    0.6279   -0.0000    0.0       0.0
  4     0.0      0.0       0.0      -0.5263    0.6016   -0.4771   -0.0000    0.0       0.0
  5     0.0      0.0       0.0      -0.1872    0.0632    0.0217   -0.0000    0.0       0.0
```

```
TRANSFER MATRIX METHOD
      X COORD(IN)         Y COORD(IN)   MODULUS OF ELASTICITY(PSI)   SPECIFIC WT(LB/CU IN)   AREA(SQ IN)        MOMENT(IN 4)
        0.0                   0.0
                                          30000000.00000               0.28300            5.88000            82.50000
        0.0                 60.00000
                                          30000000.00000               0.28300            5.88000            82.50000
        0.0                120.00000
                                          30000000.00000               0.28300           29.40000           625.00000
     180.00000             156.00000
                                          30000000.00000               0.28300           29.40000           625.00000
     360.00000             192.00000
                                          30000000.00000               0.28300           29.40000           625.00000
     540.00000             156.00000
                                          30000000.00000               0.28300           29.40000           625.00000
     720.00000             120.00000
                                          30000000.00000               0.28300            4.62000            30.30000
     720.00000              60.00000
                                          30000000.00000               0.28300            4.62000            30.30000
     720.00000               0.0
CLAMPED SIMPLE END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
                1              2               3               4               5
  1          14.765         34.238          75.052         123.234         152.589

NORMALIZED X DISPLACEMENTS
            1         2         3         4         5         6         7         8         9
  1     0.0     -0.0506   -0.1537   -0.2088   -0.2302   -0.2538   -0.3067   -0.4073   -0.5108
  2     0.0      0.1953    0.4069    0.3781    0.3217    0.3200    0.2366    0.0658   -0.1062
  3     0.0      0.0839    0.0960    0.0186    0.0800   -0.0027    0.0641    0.3640    0.7571
  4     0.0      0.0108    0.0082   -0.0011    0.0153    0.0191    0.0225   -0.3178   -0.9455
  5     0.0      0.0236   -0.0563   -0.1031   -0.0016    0.1011    0.0531    0.2574    0.8480

NORMALIZED Y DISPLACEMENTS
            1         2         3         4         5         6         7         8         9
  1     0.0      0.0000    0.0000    0.2754    0.3823    0.2644    0.0000    0.0000    0.0000
  2     0.0     -0.0000   -0.0000    0.1438    0.4256    0.4172   -0.0000   -0.0000   -0.0000
  3     0.0     -0.0000   -0.0000    0.3866    0.0797   -0.3337    0.0000    0.0000    0.0000
  4     0.0     -0.0000   -0.0000    0.0463   -0.0357   -0.0171    0.0000    0.0000    0.0000
  5     0.0     -0.0000    0.0000    0.2339   -0.2734    0.2403    0.0000    0.0000    0.0000
```

```
LUMPED PARAMETER METHOD
      X COORD(IN)         Y COORD(IN)        NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)        MOMENT(IN 4)
        0.0                   0.0
                                              6.00000              0.28300             5.88000            82.50000
        0.0                120.00000
                                             13.00000              0.28300            29.40000           625.00000
     360.00000             192.00000
                                             13.00000              0.28300            29.40000           625.00000
     720.00000             120.00000
                                              6.00000              0.28300             4.62000            30.30000
     720.00000               0.0
CLAMPED SIMPLE END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
                1              2               3               4               5
  1          14.769         34.482          74.760         123.969         150.914

NORMALIZED X DISPLACEMENTS
            1         2         3         4         5         6         7         8         9
  1     0.0      0.0500    0.1526    0.2079    0.2293    0.2528    0.3060    0.4070    0.5108
  2     0.0      0.1943    0.4067    0.3793    0.3222    0.3216    0.2377    0.0654   -0.1080
  3     0.0     -0.0854   -0.0992   -0.0210   -0.0827    0.0010   -0.0662   -0.3639   -0.7526
  4     0.0      0.0110    0.0075   -0.0022    0.0152    0.0201    0.0229   -0.3174   -0.9454
  5     0.0     -0.0213    0.0507    0.0930    0.0022   -0.0904   -0.0463   -0.2657   -0.8690

NORMALIZED Y DISPLACEMENTS
            1         2         3         4         5         6         7         8         9
  1     0.0       0.0       0.0     -0.2767   -0.3835   -0.2658    0.0000    0.0000    0.0000
  2     0.0       0.0       0.0      0.1371    0.4225    0.4195   -0.0000   -0.0000   -0.0000
  3     0.0       0.0       0.0     -0.3906   -0.0823    0.3363   -0.0000   -0.0000   -0.0000
  4     0.0       0.0       0.0      0.0488   -0.0385   -0.0140    0.0000   -0.0000   -0.0000
  5     0.0       0.0       0.0     -0.2117    0.2423   -0.2208   -0.0000   -0.0000   -0.0000
```

```
FINITE ELEMENT METHOD
      X COORD(IN)         Y COORD(IN)        NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)        MOMENT(IN 4)
        0.0                   0.0
                                              2.00000              0.28300             5.88000            82.50000
        0.0                120.00000
                                              4.00000              0.28300            29.40000           625.00000
     360.00000             192.00000
                                              4.00000              0.28300            29.40000           625.00000
     720.00000             120.00000
                                              2.00000              0.28300             4.62000            30.30000
     720.00000               0.0
CLAMPED SIMPLE END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
                1              2               3               4               5
  1          14.764         34.236          75.065         123.283         152.752

NORMALIZED X DISPLACEMENTS
            1         2         3         4         5         6         7         8         9
  1     0.0     -0.0506   -0.1537   -0.2088   -0.2302   -0.2538   -0.3067   -0.4073   -0.5108
  2     0.0      0.1953    0.4068    0.3781    0.3217    0.3200    0.2366    0.0657   -0.1062
  3     0.0     -0.0839   -0.0959   -0.0186   -0.0800    0.0027   -0.0640   -0.3640   -0.7571
  4     0.0     -0.0108   -0.0082    0.0011   -0.0153   -0.0190   -0.0225    0.3178    0.9455
  5     0.0      0.0236   -0.0564   -0.1033   -0.0016    0.1013    0.0532    0.2572    0.8476

NORMALIZED Y DISPLACEMENTS
            1         2         3         4         5         6         7         8         9
  1     0.0       0.0       0.0      0.2754    0.3823    0.2644   -0.0000    0.0       0.0
  2     0.0       0.0       0.0      0.1438    0.4256    0.4172    0.0000    0.0       0.0
  3     0.0       0.0       0.0     -0.3866   -0.0797    0.3337   -0.0000    0.0       0.0
  4     0.0       0.0       0.0     -0.0463    0.0357    0.0171   -0.0000    0.0       0.0
  5     0.0       0.0       0.0      0.2345   -0.2739    0.2407    0.0000    0.0       0.0
```

```
TRANSFER MATRIX METHOD
        X COORD(IN)          Y COORD(IN)  MODULUS OF ELASTICITY(PSI)   SPECIFIC WT(LB/CU IN)   AREA(SQ IN)         MOMENT(IN 4)
            0.0                  0.0
                                             30000000.00000                 0.28300             5.88000              82.50000
            0.0              60.00000
                                             30000000.00000                 0.28300             5.88000              82.50000
            0.0             120.00000
                                             30000000.00000                 0.28300            29.40000             625.00000
        180.00000           156.00000
                                             30000000.00000                 0.28300            29.40000             625.00000
        360.00000           192.00000
                                             30000000.00000                 0.28300            29.40000             625.00000
        540.00000           156.00000
                                             30000000.00000                 0.28300            29.40000             625.00000
        720.00000           120.00000
                                             30000000.00000                 0.28300             4.62000              30.30000
        720.00000            60.00000
                                             30000000.00000                 0.28300             4.62000              30.30000
        720.00000             0.0
PINNED PINNED END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
                 1                2                 3                4                 5
  1           15.257           25.365            75.652          149.662           277.069

NORMALIZED X DISPLACEMENTS
             1         2         3         4         5         6         7         8         9
  1      0.0       0.2170    0.3622    0.3957    0.3960    0.4152    0.4299    0.2982   -0.0000
  2      0.0       0.2132    0.2569    0.1777    0.1257    0.0927   -0.0054   -0.0689    0.0000
  3      0.0       0.1993    0.1320   -0.0020    0.1233   -0.0153    0.1145    0.2040   -0.0000
  4      0.0       0.0961   -0.1133   -0.2159    0.0078    0.2238    0.1289   -0.0993   -0.0000
  5      0.0       0.6120   -0.0077   -0.0229   -0.0013    0.0061    0.0052    0.7861    0.0000

NORMALIZED Y DISPLACEMENTS
             1         2         3         4         5         6         7         8         9
  1      0.0      -0.0000   -0.0000   -0.1678   -0.1693   -0.0733   -0.0000   -0.0000   -0.0000
  2      0.0      -0.0000   -0.0000    0.3961    0.6559    0.4906    0.0000    0.0000    0.0000
  3      0.0      -0.0000   -0.0000    0.6702    0.0438   -0.6493   -0.0000   -0.0000   -0.0000
  4      0.0      -0.0000    0.0000    0.5130   -0.6055    0.4743   -0.0000   -0.0000   -0.0000
  5      0.0      -0.0000    0.0000    0.0758   -0.0323    0.0046    0.0000    0.0000    0.0000
```

```
LUMPED PARAMETER METHOD
        X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS   SPECIFIC WT(LB/CU IN)   AREA(SQ IN)         MOMENT(IN 4)
            0.0                  0.0
                                                       6.00000                 0.28300             5.88000              82.50000
            0.0             120.00000
                                                      13.00000                 0.28300            29.40000             625.00000
        360.00000           192.00000
                                                      13.00000                 0.28300            29.40000             625.00000
        720.00000           120.00000
                                                       6.00000                 0.28300             4.62000              30.30000
        720.00000             0.0
PINNED PINNED END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
                 1                2                 3                4                 5
  I           15.285           25.365            75.162          147.574           269.188

NORMALIZED X DISPLACEMENTS
             1         2         3         4         5         6         7         8         9
  1      0.0      -0.2169   -0.3618   -0.3956   -0.3958   -0.4151   -0.4297   -0.2987    0.0000
  2      0.0      -0.2140   -0.2574   -0.1783   -0.1264   -0.0936    0.0046    0.0689    0.0000
  3      0.0      -0.2003   -0.1327    0.0013   -0.1238    0.0148   -0.1150   -0.2048    0.0000
  4      0.0      -0.0991    0.1116    0.2152   -0.0081   -0.2234   -0.1279    0.1023    0.0000
  5      0.0      -0.6146    0.0081    0.0243    0.0015   -0.0062   -0.0052   -0.7835    0.0000

NORMALIZED Y DISPLACEMENTS
             1         2         3         4         5         6         7         8         9
  1      0.0       0.0       0.0       0.1689    0.1698    0.0733   -0.0000   -0.0000   -0.0000
  2      0.0       0.0       0.0      -0.3953   -0.6550   -0.4913    0.0000    0.0000    0.0000
  3      0.0       0.0       0.0      -0.6698   -0.0444    0.6488   -0.0000   -0.0000   -0.0000
  4      0.0       0.0       0.0      -0.5178    0.5988   -0.4775    0.0000    0.0000    0.0000
  5      0.0       0.0       0.0      -0.0807    0.0332   -0.0054   -0.0000   -0.0000   -0.0000
```

```
FINITE ELEMENT METHOD
        X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS   SPECIFIC WT(LB/CU IN)   AREA(SQ IN)         MOMENT(IN 4)
            0.0                  0.0
                                                       2.00000                 0.28300             5.88000              82.50000
            0.0             120.00000
                                                       4.00000                 0.28300            29.40000             625.00000
        360.00000           192.00000
                                                       4.00000                 0.28300            29.40000             625.00000
        720.00000           120.00000
                                                       2.00000                 0.28300             4.62000              30.30000
        720.00000             0.0
PINNED PINNED END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
                 1                2                 3                4                 5
  1           15.256           25.364            75.667          149.820           278.160

NORMALIZED X DISPLACEMENTS
             1         2         3         4         5         6         7         8         9
  1      0.0       0.2170    0.3622    0.3957    0.3960    0.4152    0.4299    0.2982    0.0
  2      0.0       0.2132    0.2569    0.1777    0.1257    0.0927   -0.0054   -0.0689    0.0
  3      0.0       0.1993    0.1319   -0.0021    0.1232   -0.0154    0.1144    0.2040    0.0
  4      0.0      -0.0961    0.1133    0.2159   -0.0078   -0.2238   -0.1289    0.0993    0.0
  5      0.0      -0.6119    0.0078    0.0229    0.0013   -0.0060   -0.0051   -0.7862    0.0

NORMALIZED Y DISPLACEMENTS
             1         2         3         4         5         6         7         8         9
  1      0.0       0.0       0.0      -0.1678   -0.1693   -0.0733    0.0000    0.0       0.0
  2      0.0       0.0       0.0       0.3961    0.6559    0.4906   -0.0000    0.0       0.0
  3      0.0       0.0       0.0       0.6702    0.0438   -0.6493    0.0000    0.0       0.0
  4      0.0       0.0       0.0      -0.5130    0.6054   -0.4743   -0.0000    0.0       0.0
  5      0.0       0.0       0.0      -0.0756    0.0322   -0.0046   -0.0000    0.0       0.0
```

```
TRANSFER MATRIX METHOD
       X COORD(IN)         Y COORD(IN)  MODULUS OF ELASTICITY(PSI)  SPECIFIC WT(LB/CU IN)  AREA(SQ IN)         MOMENT(IN 4)
          0.0                  0.0
                                           30000000.00000               0.28300              5.88000              82.50000
          0.0               60.00000
                                           30000000.00000               0.28300              5.88000              82.50000
          0.0              120.00000
                                           30000000.00000               0.28300             29.40000             625.00000
       180.00000           156.00000
                                           30000000.00000               0.28300             29.40000             625.00000
       360.00000           192.00000
                                           30000000.00000               0.28300             29.40000             625.00000
       540.00000           156.00000
                                           30000000.00000               0.28300             29.40000             625.00000
       720.00000           120.00000
                                           30000000.00000               0.28300              4.62000              30.30000
       720.00000            60.00000
                                           30000000.00000               0.28300              4.62000              30.30000
       720.00000             0.0
PINNED SIMPLE END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)

                 1              2                3              4               5
  1           9.742          24.419           72.489         122.907         152.252

NORMALIZED X DISPLACEMENTS

             1        2         3         4         5         6         7         8         9
  1     0.0     -0.1556   -0.2682   -0.3028   -0.3120   -0.3270   -0.3558   -0.4092   -0.4638
  2     0.0      0.2354    0.3043    0.2398    0.1903    0.1668    0.0763   -0.1016   -0.2830
  3     0.0      0.1120    0.0700   -0.0105    0.0584   -0.0269    0.0469    0.3520    0.7421
  4     0.0      0.0152    0.0061   -0.0027    0.0137    0.0183    0.0213   -0.3189   -0.9452
  5     0.0      0.0377   -0.0589   -0.1035   -0.0051    0.0961    0.0487    0.2577    0.8532

NORMALIZED Y DISPLACEMENTS

             1        2         3         4         5         6         7         8         9
  1     0.0      0.0000    0.0000    0.1732    0.2190    0.1439    0.0000    0.0000    0.0000
  2     0.0     -0.0000   -0.0000    0.3227    0.5701    0.4527   -0.0000   -0.0000   -0.0000
  3     0.0     -0.0000   -0.0000    0.4026    0.0578   -0.3690    0.0000    0.0000    0.0000
  4     0.0     -0.0000   -0.0000    0.0444   -0.0379   -0.0151    0.0000    0.0000    0.0000
  5     0.0     -0.0000    0.0000    0.2230   -0.2689    0.2375    0.0000    0.0000    0.0000

                                          _________________________________________
LUMPED PARAMETER METHOD
       X COORD(IN)         Y COORD(IN)           NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)       MOMENT(IN 4)
          0.0                  0.0
                                                     6.00000              0.28300              5.88000           82.50000
          0.0              120.00000
                                                    13.00000              0.28300             29.40000          625.00000
       360.00000           192.00000
                                                    13.00000              0.28300             29.40000          625.00000
       720.00000           120.00000
                                                     6.00000              0.28300              4.62000           30.30000
       720.00000             0.0
PINNED SIMPLE END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)

                 1              2                3              4               5
  1           9.747          24.419           72.041         123.511         150.555

NORMALIZED X DISPLACEMENTS

             1        2         3         4         5         6         7         8         9
  1     0.0      0.1555    0.2679    0.3026    0.3118    0.3268    0.3557    0.4092    0.4639
  2     0.0     -0.2361   -0.3049   -0.2406   -0.1911   -0.1680   -0.0773    0.1007    0.2822
  3     0.0      0.1140    0.0714   -0.0101    0.0597   -0.0268    0.0479    0.3512    0.7370
  4     0.0      0.0156    0.0054   -0.0040    0.0135    0.0193    0.0217   -0.3186   -0.9451
  5     0.0     -0.0344    0.0531    0.0933    0.0056   -0.0855   -0.0419   -0.2659   -0.8737

NORMALIZED Y DISPLACEMENTS

             1         2         3         4         5         6         7         8         9
  1     0.0       0.0       0.0      -0.1740   -0.2195   -0.1444    0.0000    0.0000    0.0000
  2     0.0       0.0       0.0      -0.3216   -0.5691   -0.4534    0.0000    0.0000    0.0000
  3     0.0       0.0       0.0       0.4073    0.0587   -0.3734    0.0000    0.0000    0.0000
  4     0.0       0.0       0.0       0.0470   -0.0409   -0.0118    0.0000   -0.0000   -0.0000
  5     0.0       0.0       0.0      -0.2006    0.2376   -0.2178    0.0000    0.0000   -0.0000
                                          _________________________________________
FINITE ELEMENT METHOD
       X COORD(IN)         Y COORD(IN)           NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)       MOMENT(IN 4)
          0.0                  0.0
                                                     2.00000              0.28300              5.88000           82.50000
          0.0              120.00000
                                                     4.00000              0.28300             29.40000          625.00000
       360.00000           192.00000
                                                     4.00000              0.28300             29.40000          625.00000
       720.00000           120.00000
                                                     2.00000              0.28300              4.62000           30.30000
       720.00000             0.0
PINNED SIMPLE END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)

                 1              2                3              4               5
  1           9.741          24.417           72.502         122.956         152.412

NORMALIZED X DISPLACEMENTS

             1        2         3         4         5         6         7         8         9
  1     0.0      0.1556    0.2682    0.3028    0.3120    0.3270    0.3558    0.4092    0.4638
  2     0.0      0.2354    0.3043    0.2398    0.1903    0.1668    0.0763   -0.1017   -0.2830
  3     0.0     -0.1119   -0.0699    0.0106   -0.0584    0.0270   -0.0468   -0.3520   -0.7421
  4     0.0     -0.0152   -0.0061    0.0027   -0.0137   -0.0182   -0.0213    0.3189    0.9452
  5     0.0     -0.0378    0.0590    0.1037    0.0051   -0.0964   -0.0488   -0.2574   -0.8528

NORMALIZED Y DISPLACEMENTS

             1         2         3         4         5         6         7         8         9
  1     0.0       0.0       0.0      -0.1732   -0.2190   -0.1439    0.0000    0.0       0.0
  2     0.0       0.0       0.0       0.3227    0.5702    0.4527    0.0000    0.0       0.0
  3     0.0       0.0       0.0      -0.4025   -0.0578    0.3690   -0.0000    0.0       0.0
  4     0.0       0.0       0.0      -0.0444    0.0378    0.0152   -0.0000    0.0       0.0
  5     0.0       0.0       0.0      -0.2235    0.2694   -0.2379   -0.0000    0.0       0.0
```

```
FINITE ELEMENT METHOD
      X COORD(IN)           Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)      AREA(SO IN)         MOMENT(IN 4)
          0.0                  0.0
                                                     2.00000               0.28300              5.88000             82.50000
          0.0               120.00000
                                                     4.00000               0.28300             29.40000            625.00000
      360.00000             192.00000
                                                     4.00000               0.28300             29.40000            625.00000
      720.00000             120.00000
                                                     2.00000               0.28300              4.62000             30.30000
      720.00000               0.0
CLAMPED CLAMPED END CONDITION
ROTATORY INERTIA IS INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
              1              2                3              4              5
  1         25.748         34.675           79.585        150.207        279.725

NORMALIZEC X DISPLACEMENTS
              1          2         3         4         5         6         7         8         9
  1      0.0       -0.0364   -0.1475   -0.2312   -0.2577   -0.3016   -0.3679   -0.2051    0.0
  2      0.0        0.1980    0.4196    0.3989    0.3441    0.3469    0.2686    0.0933    0.0
  3      0.0        0.1605    0.1948    0.0614    0.1813    0.0423    0.1678    0.1616    0.0
  4      0.0        0.0587   -0.1076   -0.2129    0.0127    0.2284    0.1330   -0.0552    0.0
  5      0.0        0.6214    0.0225   -0.0151    0.0351    0.0434    0.0478    0.7536    0.0

NORMALIZED Y DISPLACEMENTS
              1          2         3         4         5         6         7         8         9
  1      0.0        0.0       0.0       0.4183    0.5510    0.3318   -0.0000    0.0       0.0
  2      0.0        0.0       0.0       0.1034    0.3774    0.3911    0.0000    0.0       0.0
  3      0.0        0.0       0.0       0.6670    0.0675   -0.6278    0.0000    0.0       0.0
  4      0.0        0.0       0.0       0.5264   -0.6015    0.4771    0.0000    0.0       0.0
  5      0.0        0.0       0.0       0.1878   -0.0633   -0.0219    0.0000    0.0       0.0
```

```
FINITE ELEMENT METHOD
      X COORD(IN)           Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)      AREA(SO IN)         MOMENT(IN 4)
          0.0                  0.0
                                                     2.00000               0.28300              5.88000             82.50000
          0.0               120.00000
                                                     4.00000               0.28300             29.40000            625.00000
      360.00000             192.00000
                                                     4.00000               0.28300             29.40000            625.00000
      720.00000             120.00000
                                                     2.00000               0.28300              4.62000             30.30000
      720.00000               0.0
CLAMFEC SIMPLE ENC CONDITION
ROTATORY INERTIA IS INCLUDED

NATURAL FREQUENCIES(RAC/SEC)
              1              2                3              4              5
  1         14.762         34.232           75.008        123.147        152.516

NORMALIZEC X DISPLACEMENTS
              1          2         3         4         5         6         7         8         9
  1      0.0       -0.0506   -0.1537   -0.2088   -0.2302   -0.2538   -0.3066   -0.4073   -0.5108
  2      0.0        0.1953    0.4069    0.3781    0.3218    0.3201    0.2367    0.0658   -0.1062
  3      0.0       -0.0839   -0.0960   -0.0187   -0.0800    0.0027   -0.0641   -0.3640   -0.7572
  4      0.0       -0.0108   -0.0082    0.0011   -0.0153   -0.0191   -0.0225    0.3177    0.9456
  5      0.0       -0.0236    0.0563    0.1031    0.0016   -0.1011   -0.0530   -0.2573   -0.8481

NCRMALIZEC Y DISPLACEMENTS
              1          2         3         4         5         6         7         8         9
  1      0.0        0.0       0.0       0.2754    0.3823    0.2644   -0.0000    0.0       0.0
  2      0.0        0.0       0.0       0.1437    0.4255    0.4171    0.0000    0.0       0.0
  3      0.0        0.0       0.0      -0.3866   -0.0798    0.3336   -0.0000    0.0       0.0
  4      0.0        0.0       0.0      -0.0463    0.0358    0.0171   -0.0000    0.0       0.0
  5      0.0        0.0       0.0      -0.2340    0.2733   -0.2403   -0.0000    0.0       0.0
```

```
FINITE ELEMENT METHOD
      X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)         MOMENT(IN 4)
         0.0                  0.0
                                                    2.00000              0.28300            5.88000             82.50000
         0.0              120.00000
                                                    4.00000              0.28300           29.40000            625.00000
     360.00000            192.00000
                                                    4.00000              0.28300           29.40000            625.00000
     720.00000            120.00000
                                                    2.00000              0.28300            4.62000             30.30000
     720.00000              0.0
PINNED PINNED END CONDITION
ROTATORY INERTIA IS INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
            1            2             3             4             5
  1      15.255       25.359        75.607       149.583       277.296

NORMALIZED X DISPLACEMENTS
          1          2          3          4          5          6          7          8          9
  1     0.0       0.2170     0.3621     0.3957     0.3960     0.4152     0.4299     0.2982     0.0
  2     0.0       0.2133     0.2570     0.1777     0.1258     0.0927    -0.0054    -0.0688     0.0
  3     0.0      -0.1994    -0.1320     0.0020    -0.1233     0.0154    -0.1145    -0.2040     0.0
  4     0.0      -0.0962     0.1132     0.2158    -0.0078    -0.2238    -0.1289     0.0993     0.0
  5     0.0      -0.6123     0.0077     0.0229     0.0013    -0.0061    -0.0052    -0.7859     0.0

NORMALIZED Y DISPLACEMENTS
          1          2          3          4          5          6          7          8          9
  1     0.0        0.0        0.0      -0.1678    -0.1693    -0.0733     0.0000     0.0        0.0
  2     0.0        0.0        0.0       0.3961     0.6559     0.4906    -0.0000     0.0        0.0
  3     0.0        0.0        0.0      -0.6702    -0.0438     0.6493    -0.0000     0.0        0.0
  4     0.0        0.0        0.0      -0.5131     0.6053    -0.4743    -0.0000     0.0        0.0
  5     0.0        0.0        0.0      -0.0758     0.0322    -0.0045    -0.0000     0.0        0.0
```

```
FINITE ELEMENT METHOD
      X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)    .    MOMENT(IN 4)
         0.0                  0.0
                                                    2.00000              0.28300            5.88000             82.50000
         0.0              120.00000
                                                    4.00000              0.28300           29.40000            625.00000
     360.00000            192.00000
                                                    4.00000              0.28300           29.40000            625.00000
     720.00000            120.00000
                                                    2.00000              0.28300            4.62000             30.30000
     720.00000              0.0
PINNED SIMPLE END CONDITION
ROTATORY INERTIA IS INCLUDED

NATURAL FREQUENCIES(RAD/SEC)
            1            2             3             4             5
  1       9.741       24.413        72.445       122.819       152.177

NORMALIZED X DISPLACEMENTS
          1          2          3          4          5          6          7          8          9
  1     0.0       0.1556     0.2682     0.3028     0.3120     0.3270     0.3559     0.4092     0.4638
  2     0.0       0.2354     0.3044     0.2399     0.1904     0.1669     0.0763    -0.1016    -0.2830
  3     0.0      -0.1120    -0.0700     0.0105    -0.0584     0.0269    -0.0469    -0.3520    -0.7421
  4     0.0      -0.0152    -0.0061     0.0028    -0.0137    -0.0183    -0.0213     0.3188     0.9452
  5     0.0      -0.0378     0.0589     0.1035     0.0051    -0.0961    -0.0486    -0.2575    -0.8533

NORMALIZED Y DISPLACEMENTS
          1          2          3          4          5          6          7          8          9
  1     0.0        0.0        0.0      -0.1732    -0.2190    -0.1439     0.0000     0.0        0.0
  2     0.0        0.0        0.0       0.3227     0.5701     0.4527     0.0000     0.0        0.0
  3     0.0        0.0        0.0      -0.4026    -0.0578     0.3690    -0.0000     0.0        0.0
  4     0.0        0.0        0.0      -0.0445     0.0379     0.0151    -0.0000     0.0        0.0
  5     0.0        0.0        0.0      -0.2230     0.2688    -0.2375    -0.0000     0.0        0.0
```

APPENDIX B

LUMPED-PARAMETER PROGRAM

General

Program LUMPED is a double precision IBM FORTRAN IV language[7] digital computer program designed to find the principal vibrational modes of straight sided plane frames of uniform cross section. The program is written for the IBM 360/67 facility installed at the Naval Postgraduate School. The in-plane undamped natural frequencies, as well as normalized displacements, may be found through its use. The displacements are calculated for the mid-point and ends of each straight member. The analysis is performed by the characterization of the structure as a chain of pin-connected rigid bars with both mass and flexibility lumped at the hinges. The flexibility is represented by torsional springs. The model neglects axial and shear deformation, but rotatory inertia can be included if the user desires.

The program is dimensioned so that a structure with up to nine straight members may be analyzed. These members may be divided into a maximum of 79 segments (total). The program with this dimensioning requires total storage of 400K bytes.

The more severely constrained end of the structure is designated the left end. This end must be either fixed or pinned. The right hand end is allowed any of the four common boundary conditions: fixed, pinned, simply supported, or free. The structure may not have any intermediate supports or externally applied forces.

Program Structure

The program consists of a main body and two subroutines. If double precision sine, cosine, arctangent, absolute value, and square root routines are not available in the FORTRAN library, they must be provided by other means since they are required by the program. The main program contains comment cards explaining each major section's function, and a large leading section giving input format information.

When the end conditions are applied, the order of the matrices is reduced. Rather than rearrange the storage of these matrices, the rows and columns corresponding to the no-longer-needed equations and rotations are replaced with zeroes. In the mass matrix, each diagonal element corresponding to a deleted row and column is replaced with unity. The eigenvalue problem may now be undertaken. The final spectral matrix will contain zeroes for the eigenvalues of the deleted rows.

The subroutine JACVAT obtains a solution for the real eigenvalue problem of a symmetric matrix. The subroutine is based on Jacobi's diagonalization by successive rotations.[8] The subroutine will take a matrix up to the maximum size of 160 by 160. Care must be taken to insure that the entering matrix is symmetric. If any of the off-diagonal elements are non-symmetric, the subroutine prints the location of these elements, and then makes them symmetric by averaging them. This output will be printed in the midst of the regular output, and is therefore not desirable.

The subroutine MAPRIN is the output subroutine. This subroutine will print a matrix, column by column. It numbers the rows and columns and allows a single line heading of 32 spaces. The heading must be in a DATA statement in the calling program. The subroutine has six different formats which may be used.

Input

For program input, the following things are needed:

1. Coordinates of the ends and the corners of the structure. A Cartesian coordinate system is used, with the positive x direction to the right and the positive y direction up. All coordinates are in inches, and start with the more severely constrained (left) end.

2. Number of segments into which each member is to be divided.

3. Properties of each member.
    a. specific weight, lb/cu.in.
    b. area, sq.in.
    c. modulus of elasticity, psi.
    d. second moment of the cross sectional area with respect to the z axis, $in.^4$

4. End conditions.

5. Angle at which right hand end condition is applied, from +90° to -90°. This angle is of concern only when the right end is simply supported. For all other cases, it is

taken as zero. The angle is the acute angle between the y axis and the line of action of the restraining force. It is measured from the y axis and is positive counterclockwise.

The FORMAT for input data, as well as the make-up of the data deck, are listed in the comment section at the start of the main program.

Output

The program will automatically print for each problem and set of end conditions within a problem the following:

1. Coordinates of the structure and number of segments between each set of coordinates.
2. The properties of each member. These properties are the same as those in the input section.
3. End conditions.
4. Whether or not rotatory inertia has been included.

The user has control over the rest of the output. The user specifies the number of natural frequencies desired with card number 220 in the program listing. The total number of frequencies requested may not exceed the number of degrees of freedom of the model minus the number of constraints. The specified number of frequencies will be printed in ascending order. The mode shapes corresponding to the frequencies are printed in two arrays. The u components are printed first and the v components second. Each row of these arrays corresponds to a particular frequency. The frequencies increase going down the columns of the arrays. Each mode shape is normalized such that the square root of the sum of the squares is unity.

```
C-----INSTRUCTIONS FOR USE OF LUMPED PARAMETER PROGRAM
C-----CONTROL CARD
C-----    FIELD I5        NUMBER OF PROBLEMS TO BE WORKED
C-----DATA CARD 1
C-----    FIELD I5        NUMBER OF SETS OF COORDINATES, N
C-----    FIELD I5        NUMBER OF END CONDITIONS, NEND
C-----    FIELD I5        ROTATORY INERTIA
C-----                              1    NOT INCLUDED
C-----                              2    INCLUDED
C-----    FIELD I5        STRAIGHT SECTION OPTION, KTROL
C-----                              1    THERE ARE CORNERS IN THE STRUCTURE
C-----                              2    THERE ARE NO CORNERS
C-----DATA CARD 2 TO N+1
C-----    FIELD 3F15.0    COORDINATES X, Y, AND NUMBER OF SEGMENTS
C-----                     BETWEEN EACH SET.  LAST CARD, SECTIONS = 0.0.
C-----DATA CARD N+2 TO 2N
C-----    FIELD 4F20.0     DENSITY, AREA, MOMENT OF INERTIA ABOUT X AXIS,
C-----                      MODULUS OF ELASTICITY FOR EACH SECTION.
C-----DATA CARD 2N+1 TO 2N+NEND
C-----    FIELD F20.0     ANGLE AT WHICH RIGHT HAND END CONDITIONS ARE APP
C-----    FIELD I5        END CONDITION
C-----                              1    CLAMPED - FREE
C-----                              2    CLAMPED - SIMPLE
C-----                              3    CLAMPED - PINNED
C-----                              4    CLAMPED - CLAMPED
C-----                              5    PINNED - FREE
C-----                              6    PINNED - SIMPLE
C-----                              7    PINNED - PINNED
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION A(10), XX(81), YY(81), SL(10), SEC(11), XM(81), PM(81),
     1          B(10), C(81), X(80,80), Y(80,80), RL(80,80), ZMX(80,80),
     2          ZMY(80,80), ZJJ(80,80), PML(80,80), XP(80), YP(80),
     3          FREQ(80), HERTZ(80), EVEC(80,80), EIVEC(80,80),
     4          XDISP(80,80), YDISP(80,80), SHAPE(4), DISPX(4), DEN(10),
     5          AR(10), XIX(10), E(10), AA(10), BB(10), DISPY(4)
      DIMENSION P(80,5), P2(80,5), KROW(4)
      EQUIVALENCE (RL(1,1), XDISP(1,1)), (ZMY(1,1), YDISP(1,1)),
     1            (XP(1), XM(1)), (YP(1), PM(1)), (EVEC(1,1), ZMX(1,1)),
     2            (EIVEC(1,1), ZJJ(1,1)), (PML(1,6), P2(1,1)),
     3            (PML(1,1), P(1,1))
      DATA SHAPE /' NATURAL FREQUENCIES(RAD/SEC) '/,
     1     DISPX /' NORMALIZED X DISPLACEMENTS  '/,
     2     DISPY /' NORMALIZED Y DISPLACEMENTS  '/
      READ (5,1) NDATA
    1 FORMAT (I5)
```

```
      DO 20 IIII=1,NDATA
      READ (5,100) N, NEND, INERT, KTROL
  100 FORMAT (4I5)
      READ (5, 101) (A(I), B(I), SEC(I), I=1, N)
  101 FORMAT (3F15.0)
      NN = N - 1
      READ (5,104) (DEN(I), AR(I), XIX(I), E(I), I=1,NN)
  104 FORMAT (4F20.0)
      DO 20  NUM=1, NEND
      READ(5,109) PHI, II
  109 FORMAT (F20.0, I5)
      WRITE (6,106)
  106 FORMAT (1H1, 24H LUMPED PARAMETER METHOD)
      WRITE (6,105)
  105 FORMAT (1H0, 8X, 11HX COORD(IN), 9X, 11HY COORD(IN), 7X, 18HNUMBER
     1 OF SECTIONS, 2X, 21HSPECIFIC WT(LB/CU IN), 4X, 11HAREA(SQ IN),
     2 6X, 12HMOMENT(IN 4))
      WRITE (6,107) (A(I), B(I), SEC(I), DEN(I), AR(I), XIX(I), I=1,NN)
  107 FORMAT (2F20.5, /, 40X, 4F20.5)
      WRITE (6,400) A(N), B(N)
  400 FORMAT (2F20.5)
  220 LIM = 5
      AA(1) = A(1)
      BB(1) = B(1)
      XX(1) =AA(1)
      YY(1) =BB(1)
  406 M = 1
C-----
C-----ROTATION OF AXIS IF PHI NOT EQUAL TO ZERO
C-----
      PI = 3.14159265358979
      IF (PHI) 92, 401, 92
  401 DO 402 I=2,N
      AA(I) = A(I)
  402 BB(I) = B(I)
      GO TO 94
   92 PHI = PHI * PI / 0.180D3
      DO 200 I=2,N
      IF (A(I) - A(1)) 96, 93, 96
   93 THETA = PI * 0.5D0
      IF (B(I).LT.B(1)) THETA = -THETA
      GO TO 160
   96 ARG = (B(I) - B(1)) / (A(I) - A(1))
      THETA = DATAN (ARG)
  160 BETA = THETA - PHI
```

```
      DIST = DSQRT ((B(I) - B(1))**2 + (A(I) - A(1))**2)
      BB(I)= DIST * DSIN (BETA)
  200 AA(I)= DIST * DCOS (BETA)
C-----
C-----GENERATION OF MASS AND STIFFNESS MATRICES
C-----
   94 DO 10 I = 1, NN
      K = SEC(I)
      XL = DSQRT ((AA(I+1)-AA(I)) **2 + (BB(I+1)-BB(I)) **2)
      SL(I) = XL / SEC(I)
      DO 11 J = 1, K
      ZJ = J
      W = DEN(I) * AR(I) / 0.386D3
      Z = (ZJ - 0.50D0) / SEC(I)
      XX(M+J) = (0.1D1 - Z) *AA(I) + Z *AA(I+1)
      YY(M+J) = (0.1D1 - Z) *BB(I) + Z *BB(I+1)
      XM(M+J) = SL(I) * W
      IF (INERT.EQ.1) GO TO 103
      PM(M+J) = XM(M+J) * ((SL(I)**2) / 0.12D2 + XIX(I) / AR(I))
      GO TO 11
  103 PM(M+J) = XM(M+J) * (SL(I)**2) / 0.12D2
   11 C(M+J) = E(I) * XIX(I) / SL(I)
   10 M = M + J
      XX(M+1) =AA(N)
      YY(M+1) =BB(N)
      DO 12 J= 1, M
      DO 12 I = 1, M
      IF (I-J) 13, 14, 14
   13 Y(I,J) = 0.0
      X(I,J) = 0.0
      RL(I,J) = 0.0
      GO TO 12
   14 Y(I,J) = YY(I+1) - YY(J)
       X(I,J) = XX(I+1) - XX(J)
      IF (DABS(Y(I,J)).LT.0.1D-9) Y(I,J)=0.0
      IF (DABS(X(I,J)).LT.0.1D-9) X(I,J)=0.0
      RL(I,J) = 1.0
   12 CONTINUE
      C(1) = 0.0
      PM(1) = 0.0
      PM(M+1) = 0.0
      XM(M+1) = 0.0
      DO 15 I=1, M
      PM(I) = 0.5 * (PM(I) + PM(I+1))
   15 XM(I) = XM(I+1)
```

```
      DO 16 I=1, M
      DO 16 J=1, M
      ZMX(I,J) = XM(I) * X(I,J)
      ZMY(I,J) = XM(I) * Y(I,J)
   16 PML(I,J) = PM(I) * RL(I,J)
      DO 19 L=1,M
      DO 19 I=1,M
      ZJJ(I,L) = 0.0
      DO 19 J=1,M
   19 ZJJ(I,L) = ZJJ(I,L) + X(J,I) * ZMX(J,L) + Y(J,I) * ZMY(J,L) +
     1 RL(J,I) * PML(J,L)
      KZ = M + 1
      GO TO (73, 74, 75, 76, 77, 78, 79), II
   73 WRITE (6,110)
  110 FORMAT (1HO, 27H CLAMPED FREE END CONDITION)
      GO TO 80
   74 WRITE (6,111)
  111 FORMAT (1HO, 29H CLAMPED SIMPLE END CONDITION)
      GO TO 80
   75 WRITE (6,112)
  112 FORMAT (1HO, 29H CLAMPED PINNED END CONDITION)
      GO TO 80
   76 WRITE (6,113)
  113 FORMAT (1HO, 30H CLAMPED CLAMPED END CONDITION)
      GO TO 80
   77 WRITE (6,114)
  114 FORMAT (1HO, 26H PINNED FREE END CONDITION)
      GO TO 80
   78 WRITE (6,115)
  115 FORMAT (1HO, 28H PINNED SIMPLE END CONDITION)
      GO TO 80
   79 WRITE (6,116)
  116 FORMAT (1HO, 28H PINNED PINNED END CONDITION)
   80 IF (INERT.NE.1) GO TO 225
      WRITE (6,226)
  226 FORMAT (1HO, 33H ROTATORY INERTIA IS NOT INCLUDED)
      GO TO 227
  225 WRITE (6,228)
  228 FORMAT (1HO, 29H ROTATORY INERTIA IS INCLUDED)
  227 CONTINUE
C-----
C-----GENERATION OF END CONDITION MATRIX
C-----
      DO 21 I=1,M
      P2(I,1) = RL(1,I)
      P2(I,2) = X(M,I)
      P2(I,3) = - Y(M,I)
   21 P2(I,4) = RL(M,I)
      DO 17 I=1,M
      DO 18 J=1,M
   18 RL(I,J) = 0.0
   17 RL(I,I) = C(I)
      IF (II.GE.5) GO TO 23
      KST = 0
   22 GO TO (24, 25, 26, 27, 28, 24, 25, 26), II
   23 KST = 1
      GO TO 22
   24 KEND = 1
      GO TO 29
   25 KEND = 2
      GO TO 29
   26 KEND = 3
      GO TO 29
   27 KEND = 4
      GO TO 29
   29 KEN = KEND
      DO 30 J=1,KEND
      DO 30 I=1,M
      P(I,J) = P2(I,J+KST)
   30 P2(I,J) = P(I,J)
C-----
C-----APPLICATION OF END CONDITIONS USING A GAUSS ELIMINATION
C-----
      JZ = 1
      DO 31 KK=1,KEND
      IF (KEN.LT.KK) GO TO 31
   33 IF (P(JZ,KK).NE.0.0) GO TO 32
      JZ = JZ + 1
      IF (JZ.GT.M) GO TO 41
      GO TO 33
   32 IF(KK.EQ.1) GO TO 34
      IF (KROW(KK-1).EQ.KK-1) GO TO 34
      KROW(KK) = JZ
      JZ = KK
      GO TO 40
   34 KROW(KK) = JZ
      JZ = KK + 1
   40 JJ = KROW(KK)
      FAC = P(JJ,KK)
      DO 35 J=1,KEND
```

```
35 P(JJ,J) = P(JJ,J) / FAC
   DO 36 J=1,M
   RL(JJ,J) = RL(JJ,J) / FAC
36 ZJJ(JJ,J) = ZJJ(JJ,J) / FAC
   DO 37 I=1,M
   IF (I.EQ.JJ) GO TO 37
   FAC = P(I,KK)
   DO 38 J=1,KEND
   P(I,J) = P(I,J) - P(JJ,J) * FAC
38 IF (DABS(P(I,J)).LT.0.1D-9) P(I,J) = 0.0
   DO 39 J=1,M
   RL(I,J) = RL(I,J) - RL(JJ,J) * FAC
39 ZJJ(I,J) = ZJJ(I,J) - ZJJ(JJ,J) * FAC
37 CONTINUE
   GO TO 31
41 WRITE (6,42) KK
42 FORMAT (1H0, 28HALL ZEROES IN COLUMN NUMBER , I2)
   KEN = KEN - 1
   IF (KEN.LT.KK) GO TO 31
   DO 43 J=KK,KEN
   DO 43 I=1,M
   P(I,J) = P(I,J+1)
43 P2(I,J) = P2(I,J+1)
   JZ = KK
   GO TO 33
31 CONTINUE
   KEND = KEN
   DO 44 I=1,KEND
   JJ = KROW(I)
   DO 44 J=1,M
   ZJJ(JJ,J) = P2(J,I)
44 RL(JJ,J) = P2(J,I)
   DO 45 KK=1,KEND
   JJ = KROW(KK)
   FAC1 = ZJJ(JJ,JJ)
   FAC2 = RL(JJ,JJ)
   DO 46 I=1,M
   ZJJ(JJ,I) = ZJJ(JJ,I) / FAC1
46 RL(JJ,I) = RL(JJ,I) / FAC2
   DO 69 I=1,M
   IF (I.EQ.JJ) GO TO 69
   FAC1 = ZJJ(I,JJ)
   FAC2 = RL(I,JJ)
   DO 47 J=1,M
   ZJJ(I,J) = ZJJ(I,J) - ZJJ(JJ,J) * FAC1
   47 RL(I,J) = RL(I,J) - RL(JJ,J) * FAC2
   69 CONTINUE
   45 CONTINUE
      DO 48 I=1,KEND
      JJ = KROW(I)
      DO 48 J=1,M
      ZJJ(JJ,J) = 0.0
      ZJJ(J,JJ) = 0.0
      IF (J.EQ.JJ) ZJJ(JJ,J) = 0.1D1
      RL(J,JJ) = 0.0
   48 RL(JJ,J) = 0.0
C-----
C-----MAKING SURE OFF DIAGONAL ELEMENTS ARE EQUAL
C------
   28 DO 55 I=1,M
      DO 55 J=1,M
      ZJJ(I,J) = (ZJJ(I,J) + ZJJ(J,I)) * 0.5D0
      ZJJ(J,I) = ZJJ(I,J)
      RL(I,J) = (RL(I,J) + RL(J,I)) * 0.5D0
   55 RL(J,I) = RL(I,J)
C-----
C-----EIGENVALUE PROBLEM ON CONVERTED J MATRIX
C-----
      CALL JACVAT (ZJJ, M, 1, HERTZ, EVEC, 80)
C-----
C-----EIGENVALUE PROBLEM ON CONVERTED K MATRIX
C-----
      DO 49 I=1,M
      HERTZ(I) = DABS(HERTZ(I))
   49 HERTZ(I) = 0.1D1 / DSQRT(HERTZ(I))
      DO 50 L=1,M
      DO 50 I=1,M
      ZMY(I,L) = 0.0
      DO 50 J=1,M
   50 ZMY(I,L) = ZMY(I,L) + RL(I,J) * EVEC(J,L) * HERTZ(L)
      DO 51 L=1,M
      DO 51 I=1,M
      ZJJ(I,L) = 0.0
      DO 51 J=1,M
   51 ZJJ(I,L) = ZJJ(I,L) + HERTZ(I) * EVEC(J,I) * ZMY(J,L)
C-----
C-----MAKING SURE OFF DIAGONAL ELEMENTS ARE EQUAL
C-----
      DO 52 I=1,M
      DO 52 J=1,M
      ZJJ(I,J) = (ZJJ(I,J) + ZJJ(J,I)) * 0.5D0
   52 ZJJ(J,I) = ZJJ(I,J)
      CALL JACVAT (ZJJ, M, 1, FREQ, ZMY, 80)
      DO 53 L=1,M
      FREQ(L) = DABS(FREQ(L))
   53 FREQ(L) = DSQRT (FREQ(L))
C-----
C-----OBTAINING ROTATIONS FROM EIGENVECTOR
C-----
      DO 54 L=1,M
      DO 54 I=1,M
      EIVEC(I,L) = 0.0
      DO 54 J=1,M
   54 EIVEC(I,L) = EIVEC(I,L) + EVEC(I,J) * HERTZ(J) * ZMY(J,L)
      MM = M - KEND
      IF (LIM.GT.MM) LIM=MM
C-----
C-----SIFTING TO PUT IN ORDER OF ASCENDING FREQUENCY
C-----
      LL = LIM + 1
      IF (LL.GE.M)  LL=M-1
      DO 63 L=1,LL
      TEST = FREQ(L)
      NCOL = L
      IL = L + 1
      DO 64 I=IL,M
      IF (II.EQ.5) GO TO 82
      IF (TEST.EQ.0.0) GO TO 65
      IF (FREQ(I).EQ.0.0) GO TO 64
   82 IF (FREQ(I)-TEST) 65, 64, 64
   65 TEST = FREQ(I)
      NCOL = I
   64 CONTINUE
      DO 66 J=1,M
      EVEC(J,L) = EIVEC(J,NCOL)
      EIVEC(J,NCOL) = EIVEC(J,L)
   66 EIVEC(J,L) = EVEC(J,L)
      FREQ(NCOL) = FREQ(L)
   63 FREQ(L) = TEST
C-----
C-----RECONSTRUCTION OF ROTATION EIGENVECTORS
C-----
      IF (II.EQ.5) GO TO 72
      DO 67 KK=1,KEND
      JJ = KROW(KK)
      DO 67 J=1,M
   67 EIVEC(JJ,J) = 0.0
      DO 83 NP=1,LIM
      DO 56 KK=1,KEND
      P2(KK,5) = 0.0
      DO 56 J=1,M
   56 P2(KK,5) = P2(KK,5) - P2(J,KK) * EIVEC(J,NP)
      KEN = KEND + 1
      DO 58 KK=1,KEND
      JJ = KROW(KK)
      DO 59 I=1,KEND
   59 P(I,KK) = P2(JJ,I)
   58 P(KK,KEN) = P2(KK,5)
      DO 60 KK=1,KEND
      FAC = P(KK,KK)
      DO 61 I=1,KEN
   61 P(KK,I) = P(KK,I) / FAC
      DO 62 I=1,KEND
      IF (I.EQ.KK) GO TO 62
      FAC = P(I,KK)
      DO 62 J=1,KEN
      P(I,J) = P(I,J) - P(KK,J) * FAC
      IF(DABS(P(I,J)).LT.0.1D-9) P(I,J)=0.0
   62 CONTINUE
   60 CONTINUE
      DO 57 I=1,KEND
      JJ = KROW(I)
   57 EIVEC(JJ,NP) = P(I,KEN)
   83 CONTINUE
   72 CALL MAPRIN (FREQ, 1, LIM, 1, 2, SHAPE)
C-----
C-----CONSTRUCTION OF MATRICES FOR COMPARISON OF DISPLACEMENTS
C-----
      XP(1) =AA(1)
      YP(1) =BB(1)
      JJ = 0
      DO 205 I=2,N
      JJ = JJ + 2
      XP(JJ) =AA(I-1) + 0.5D0 * (AA(I)-AA(I-1))
      YP(JJ) =BB(I-1) + 0.5D0 * (BB(I)-BB(I-1))
      XP(JJ+1) =AA(I)
  205 YP(JJ+1) =BB(I)
      JJ = JJ + 1
      DO 201 J=1,M
      DO 201 I=1,JJ
      X(I,J) = 0.0
  201 Y(I,J) = 0.0
      TOTAL = 1.0000
      DO 202 I=1,NN
      IJ = SEC(I) / 2.0 + TOTAL
      JI = SEC(I) + TOTAL
      DO 203 J=1,IJ
       X(2*I,J) = XP(2*I) - XX(J)
  203  Y(2*I,J) = YP(2*I) - YY(J)
      DO 204 J=1,JI
      X(2*I+1,J) = XP(2*I+1) - XX(J)
  204 Y(2*I+1,J) = YP(2*I+1) - YY(J)
  202 TOTAL = JI
C-----
C-----GENERATION OF X AND Y DISPLACEMENTS
C-----
      DO 122 I=1,LIM
      DO 122 L=1,JJ
      XDISP(I,L) = 0.0
      YDISP(I,L) = 0.0
      DO 122 J=1,M
      XDISP(I,L) = XDISP(I,L) - Y(L,J) * EIVEC(J,I)
  122 YDISP(I,L) = YDISP(I,L) + X(L,J) * EIVEC(J,I)
      DO 211 I=1,LIM
      DO 211 L=1,JJ
      YN = DCOS(PHI) * YDISP(I,L) + DSIN(PHI) * XDISP(I,L)
      XN = - DSIN(PHI) * YDISP(I,L) + DCOS(PHI) * XDISP(I,L)
      YDISP(I,L) = YN
  211 XDISP(I,L) = XN
C-----
C-----NORMALIZING OF DISPLACEMENTS
C-----
      DO 210 I=1,LIM
      SUM1 = 0.0
      DO 206 J=1,JJ
  206 SUM1 = SUM1 + YDISP(I,J)**2 + XDISP(I,J)**2
      SUM = DSQRT (SUM1)
      DO 210 J=1,JJ
      XDISP(I,J) = XDISP(I,J) / SUM
  210 YDISP(I,J) = YDISP(I,J) / SUM
      CALL MAPRIN (XDISP, LIM, JJ, 80, 6, DISPX)
      CALL MAPRIN (YDISP, LIM, JJ, 80, 6, DISPY)
   20 CONTINUE
      STOP
      END
      SUBROUTINE MAPRIN (A,N,M,ND,ICODE,TITLE)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION A(ND,1), TITLE(4), NKOL(6), F(4,6), FM1(2), FM2(2),
     1 F1(4), F2(4), F3(4), F4(4), F5(4), F6(4)
      EQUIVALENCE (F1,F),(F2,F(5)),(F3,F(9)),(F4,F(13)),(F5,F(17)),
     1 (F6,F(21))
C
C     MAPRIN PRINTS 'N' ROWS AND 'M' COLUMNS OF MATRIX A (ROW
C     DIMENSION = ND) BY PARTITIONING INTO GROUPS OF 'NK' COLUMNS.
C     ICODE = 1 TO 6 SPECIFIES THE FORMAT (SEE DATA STATEMENT FOR F).
C     THIS VERSION FOR IBM 360.
C
      DATA NKOL /8,8,10,10,12,12/
      DATA  F1 /32H(/2X 8I15)       (I5,2X, 8D15.7)/,
     1      F2 /32H(/2X 8I15)       (I5,1X, 8F15.3)/,
     2      F3 /32H(/3X10I12)       (I5,2X,10D12.4)/,
     3      F4 /32H(/3X10I12)       (I5,1X,10F12.6)/,
     4      F5 /32H(/4X12I10)       (I5,2X,12D10.2)/,
     5      F6 /32H(/4X12I10)       (I5,1X,12F10.4)/
      IF (ICODE.LT.1.OR.ICODE.GT.6)  ICODE = 1
      NK = NKOL(ICODE)
      FM1(1) = F(1,ICODE)
      FM1(2) = F(2,ICODE)
      FM2(1) = F(3,ICODE)
      FM2(2) = F(4,ICODE)
      NPR = 1 + (M-1)/NK
      N1 = 1
      N2 = MINO(M,NK)
      PRINT 10, TITLE
   10 FORMAT (/1H0,4A8)
      DO 150  K = 1,NPR
      PRINT FM1, (J, J=N1,N2)
      DO 140  I = 1,N
  140 PRINT FM2, I, (A(I,J), J=N1,N2)
      N1 = N1 + NK
  150 N2 = MINO(M,N2+NK)
      RETURN
      END
      SUBROUTINE JACVAT(A,N,NOYES,EIVU,EIVR,NDIM)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION A(NDIM,NDIM),EIVU(NDIM),EIVR(NDIM,NDIM)
      IF(N-1)20,23,21
   20 PRINT 22,N
   22 FORMAT(4H N= ,I3,68H IS TOO SMALL.  LIMIT IS 1.  RETURN TO CALLING
     1 ROUTINE FROM JACVAT. )
      RETURN
   23 PRINT 24, A(1,1)
   24 FORMAT (24H IN JACVAT , MATRIX A = ,E14.6)
      RETURN
   21 IF(N-160)1,1,3
    3 PRINT 2,N
    2 FORMAT(3H N=,I5,70H IS TOO LARGE.  LIMIT IS 160.  RETURN TO CALLI-
     1NG ROUTINE FROM JACVAT. )
      RETURN
    1 IF(NOYES)99,102,99
   99 CONTINUE
      DO 101 J=1,N
      DO 100 I=1,N
  100 EIVR(I,J)=0.0
  101 EIVR(J,J)=1.0
  102 ATOP=0.
      DO 112 J=1,N
      DO 111 I=1,J
      IF(A(I,J)-A(J,I))90,103,90
   90 PRINT 106,N,N
  106 FORMAT(14H IN JACVAT (A(,I3,1H,,I3,3H)),)
      PRINT 108,I,J,J,I
  108 FORMAT(3H A(,I3,1H,,I3,10H)  AND  A(,I3,1H,,I3,54H) WERE UNEQUAL,
     1SO THEY WERE REPLACED WITH THEIR MEAN )
      A(I,J)=.5*(A(I,J)+A(J,I))
      A(J,I)=A(I,J)
  103 CONTINUE
      IF(ATOP-DABS(A(I,J)))104,111,111
  104 ATOP=DABS(A(I,J))
  111 CONTINUE
  112 EIVU(J)=A(J,J)
      IF(ATOP)109,109,113
  109 PRINT 110
  110 FORMAT(26H IN JACVAT, MATRIX A  = 0 )
```

```
      RETURN
  113 AVGF=DFLOAT(N*(N-1))*.55
      D=0.0
      DO 114  JJ=2,N
      DO 114  II=2,JJ
      S=A(II-1,JJ)/ATOP
  114 D=S*S+D
      DSTOP=(1.E-06)*D
      THRSH =  DSQRT(D/AVGF)*ATOP
  115 IFLAG=0
      DO 130  JCOL=2,N
      JCOL1=JCOL-1
      DO 130  IROW=1,JCOL1
      AIJ=A(IROW,JCOL)
      IF(DABS(AIJ)-THRSH)130,130,117
  117 AII=A(IROW,IROW)
      AJJ=A(JCOL,JCOL)
      S=AJJ-AII
      IF(DABS(AIJ)-1.E-09*DABS(S))130,130,118
  118 IFLAG=1
      IF(1.E-10*DABS(AIJ)-DABS(S))116,119,119
  119 S=.707106781
      C=S
      GO TO 120
  116 T=AIJ/S
      S=0.25/DSQRT(0.25+T*T)
      C=DSQRT(0.5+S)
      S=2.*T*S/C
  120 DO 121  I=1,IROW
      T=A(I,IROW)
      U=A(I,JCOL)
      A(I,IROW)=C*T-S*U
  121 A(I,JCOL)=S*T+C*U
      I2=IROW+2
      IF(I2-JCOL)127,127,123
  127 CONTINUE
      DO 122  I=I2,JCOL
      T=A(I-1,JCOL)
      U=A(IROW,I-1)
      A(I-1,JCOL)=S*U+C*T
  122 A(IROW,I-1)=C*U-S*T
  123 A(JCOL,JCOL)=S*AIJ+C*AJJ
      A(IROW,IROW)=C*A(IROW,IROW)-S*(C*AIJ-S*AJJ)
      DO 124  J=JCOL,N
      T=A(IROW,J)
      U=A(JCOL,J)
      A(IROW,J)=C*T-S*U
  124 A(JCOL,J)=S*T+C*U
      IF(NOYES)131,126,131
  131 CONTINUE
      DO 125 I=1,N
      T=EIVR(I,IROW)
      EIVR(I,IROW)=C*T-EIVR(I,JCOL)*S
  125 EIVR(I,JCOL)=S*T+EIVR(I,JCOL)*C
  126 CONTINUE
      S=AIJ/ATOP
      D=D-S*S
      IF(D-DSTOP)1260,129,129
 1260 D=0.
      DO 128 JJ=2,N
      DO 128 II=2,JJ
      S=A(II-1,JJ)/ATOP
  128 D=S*S+D
      DSTOP=(1.E-06)*D
  129 THRSH=DSQRT(D/AVGF)*ATOP
  130 CONTINUE
      IF(IFLAG)115,134,115
  134 T=A(1,1)
      A(1,1)=EIVU(1)
      EIVU(1)=T
      DO 132 J=2,N
      T=A(J,J)
      A(J,J)=EIVU(J)
      EIVU(J)=T
      DO 132 I=2,J
  132 A(I-1,J)=A(J,I-1)
  133 RETURN
      END
```

```
LUMPEO PARAMETER METHOO
      X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)         MOMENT(IN 4)
         0.0                  0.0
                                                    6.00000              0.28300             4.62000              30.30000
         0.0              120.00000
                                                   13.00000              0.28300            29.40000             625.00000
     360.00000            192.00000
                                                   13.00000              0.28300            29.40000             625.00000
     720.00000            120.00000
                                                    6.00000              0.28300             5.88000              82.50000
     720.00000              0.0
CLAMPEO CLAMPEO ENO CONOITION
ROTATORY INERTIA IS NOT INCLUOEO

NATURAL FREQUENCIES(RAO/SEC)

                 1                 2                   3                  4                  5
   1          25.921            34.929              79.366            148.262            271.688

NORMALIZEO X OISPLACEMENTS

              1        2          3          4          5          6          7          8          9
   1      0.0      0.2042     0.3664     0.2999     0.2558     0.2294     0.1452     0.0352     0.0000
   2      0.0      0.0934     0.2696     0.3483     0.3444     0.4000     0.4192     0.1969     0.0000
   3      0.0     -0.1623    -0.1706    -0.0455    -0.1845    -0.0653    -0.1984    -0.1613    -0.0000
   4      0.0      0.0560    -0.1323    -0.2284    -0.0134     0.2119     0.1054    -0.0600     0.0000
   5      0.0      0.7492     0.0511     0.0464     0.0379    -0.0160     0.0247     0.6208    -0.0000

NORMALIZEO Y OISPLACEMENTS

              1          2          3        4          5          6          7          8          9
   1      0.0        0.0        0.0      0.3326     0.5531     0.4208    -0.0000    -0.0000    -0.0000
   2      0.0        0.0        0.0     -0.3936    -0.3741    -0.0964    -0.0000    -0.0000    -0.0000
   3      0.0        0.0        0.0     -0.6255     0.0693     0.6654     0.0000     0.0000     0.0000
   4      0.0        0.0        0.0      0.4804    -0.5943     0.5324    -0.0000    -0.0000    -0.0000
   5      0.0        0.0        0.0      0.0232     0.0658    -0.2034     0.0000     0.0000     0.0000
```

```
LUMPEO PARAMETER METHOO
      X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)     AREA(SQ IN)         MOMENT(IN 4)
         0.0                  0.0
                                                    6.00000              0.28300             4.62000              30.30000
         0.0              120.00000
                                                   13.00000              0.28300            29.40000             625.00000
     360.00000            192.00000
                                                   13.00000              0.28300            29.40000             625.00000
     720.00000            120.00000
                                                    6.00000              0.28300             5.88000              82.50000
     720.00000              0.0
PINNEO PINNEO ENO CONOITION
ROTATORY INERTIA IS NOT INCLUOED

NATURAL FREQUENCIES(RAO/SEC)

                 1                 2                   3                  4                  5
   1          15.285            25.365              75.162            147.574            269.188

NORMALIZEO X OISPLACEMENTS

              1        2          3          4          5          6          7          8          9
   1      0.0     -0.2987    -0.4297    -0.4151    -0.3958    -0.3956    -0.3618    -0.2169     0.0000
   2      0.0      0.0689     0.0046    -0.0936    -0.1264    -0.1783    -0.2574    -0.2140    -0.0000
   3      0.0     -0.2048    -0.1150     0.0148    -0.1238     0.0013    -0.1327    -0.2003     0.0000
   4      0.0     -0.1023     0.1279     0.2234     0.0081    -0.2152    -0.1116     0.0991     0.0000
   5      0.0     -0.7835    -0.0052    -0.0062     0.0015     0.0243     0.0081    -0.6146    -0.0000

NORMALIZEO Y OISPLACEMENTS

              1          2          3        4          5          6          7          8          9
   1      0.0        0.0        0.0     -0.0733    -0.1698    -0.1689     0.0000     0.0000     0.0000
   2      0.0        0.0        0.0      0.4913     0.6550     0.3953    -0.0000    -0.0000    -0.0000
   3      0.0        0.0        0.0     -0.6488     0.0444     0.6698    -0.0000    -0.0000    -0.0000
   4      0.0        0.0        0.0     -0.4775     0.5988    -0.5178     0.0000     0.0000     0.0000
   5      0.0        0.0        0.0      0.0054    -0.0332     0.0807    -0.0000    -0.0000    -0.0000
```

```
LUMPED PARAMETER METHOD
     X COORD(IN)          Y COORD(IN)         NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)      AREA(SQ IN)          MOMENT(IN 4)
       0.0                  0.0
                                                   6.00000              0.28300                5.88000               82.50000
       0.0                120.00000
                                                  12.00000              0.28300               29.40000              625.00000
     360.00000            192.00000
                                                  12.00000              0.28300               29.40000              625.00000
     720.00000            120.00000
                                                   6.00000              0.28300                4.62000               30.30000
     720.00000              0.0
CLAMPED CLAMPED END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)

               1              2               3              4               5
  1          25.916         34.922          79.284        147.905         270.369

NORMALIZED X DISPLACEMENTS

             1          2          3          4          5          6          7          8          9
  1      0.0        0.0352     0.1452     0.2293     0.2559     0.3001     0.3666     0.2043     0.0000
  2      0.0       -0.1970    -0.4195    -0.4002    -0.3447    -0.3484    -0.2699    -0.0935    -0.0000
  3      0.0       -0.1625    -0.2001    -0.0672    -0.1861    -0.0473    -0.1721    -0.1635    -0.0000
  4      0.0       -0.0609     0.1062     0.2120    -0.0137    -0.2289    -0.1335     0.0567     0.0000
  5      0.0       -0.6215    -0.0252     0.0147    -0.0385    -0.0470    -0.0517    -0.7495    -0.0000

NORMALIZED Y DISPLACEMENTS

             1          2          3          4          5          6          7          8          9
  1      0.0        0.0        0.0       -0.4201    -0.5535    -0.3325    -0.0000    -0.0000    -0.0000
  2      0.0        0.0        0.0       -0.0967    -0.3741    -0.3924     0.0000     0.0000     0.0000
  3      0.0        0.0        0.0       -0.6644    -0.0700     0.6241    -0.0000    -0.0000    -0.0000
  4      0.0        0.0        0.0       -0.5291     0.5993    -0.4769     0.0000     0.0000     0.0000
  5      0.0        0.0        0.0       -0.1999     0.0662     0.0234    -0.0000    -0.0000     0.0000
```

```
LUMPED PARAMETER METHOD
     X COORD(IN)          Y COORD(IN)         NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)      AREA(SQ IN)          MOMENT(IN 4)
       0.0                  0.0
                                                   4.00000              0.28300                5.88000               82.50000
       0.0                120.00000
                                                  13.00000              0.28300               29.40000              625.00000
     360.00000            192.00000
                                                  13.00000              0.28300               29.40000              625.00000
     720.00000            120.00000
                                                   4.00000              0.28300                4.62000               30.30000
     720.00000              0.0
CLAMPED CLAMPED END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)

               1              2               3              4               5
  1          26.180         35.301          79.662        148.348         271.730

NORMALIZED X DISPLACEMENTS

             1          2          3          4          5          6          7          8          9
  1      0.0        0.0338     0.1427     0.2273     0.2538     0.2983     0.3650     0.2032     0.0
  2      0.0       -0.1956    -0.4189    -0.4009    -0.3446    -0.3494    -0.2703    -0.0933     0.0000
  3      0.0       -0.1617    -0.2011    -0.0683    -0.1869    -0.0480    -0.1727    -0.1627     0.0000
  4      0.0        0.0600    -0.1049    -0.2116     0.0138     0.2286     0.1326    -0.0558    -0.0000
  5      0.0       -0.6208    -0.0240     0.0174    -0.0376    -0.0467    -0.0511    -0.7480    -0.0000

NORMALIZED Y DISPLACEMENTS

             1          2          3          4          5          6          7          8          9
  1      0.0        0.0        0.0       -0.4230    -0.5558    -0.3332    -0.0000    -0.0000    -0.0000
  2      0.0        0.0        0.0       -0.0899    -0.3716    -0.3959     0.0000     0.0000     0.0000
  3      0.0        0.0        0.0       -0.6642    -0.0709     0.6237    -0.0000    -0.0000    -0.0000
  4      0.0        0.0        0.0        0.5333    -0.5937     0.4802    -0.0000    -0.0000    -0.0000
  5      0.0        0.0        0.0       -0.2072     0.0677     0.0219    -0.0000    -0.0000    -0.0000
```

APPENDIX C

FINITE ELEMENT PROGRAM

General

Program FINITE is a double precision IBM FORTRAN IV language[7] digital computer program designed to find the principal vibrational modes of straight sided plane frames of uniform cross section. The program is written for the IBM 360/67 facility installed at the Naval Postgraduate School. The program uses a finite-element approach to calculate the in-plane undamped natural frequencies and normalized displacements of structures. The program neglects axial and shear deformation, but rotatory inertia may be included if desired.

The program is dimensioned to handle a structure with a maximum of 20 members, and a maximum total of 43 elements. The program with this dimensioning requires total storage of 280K bytes. The properties of each straight member must be uniform, but may vary from member to member.

The more severely constrained end of the structure is considered the left end, and must be pinned or fixed. The right end may be fixed, pinned, simply supported, or free. The program cannot handle any intermediate supports or externally applied forces.

Program Structure

The program consists of a main body and four subroutines. The program requires the use of double precision sine, cosine, arctangent, absolute value, and square root routines. If they are not included in the FORTRAN library they must be provided by other means.

The sections of the main body and their functions are explained by comment cards in the program listing. There is also a large leading section giving input format information.

The subroutine ELEM generates the elemental stiffness and mass matrices for use in the main program.

The subroutine CORNER performs the corner transformation on the last element to the left of the corner and the first element to the right of the corner.

The subroutines MAPRIN and JACVAT are the same as used in the LUMPED program. These subroutines perform the same function in the FINITE program as they did in the LUMPED program. Once again the rows and columns of the mass and stiffness matrices deleted in applying end conditions are replaced by zeroes. In the mass matrix, each diagonal element corresponding to a deleted row and column is replaced with unity.

Input

The input required by the FINITE program is the same as that needed for the LUMPED program with one exception. The number of elements into which a member is subdivided must be an even number. The input information for the LUMPED program is found on page 58 of Appendix B.

Output

The form and type of output for the FINITE program is identical to that of the LUMPED program and is discussed on page 59 of Appendix B. In order to specify the number of frequencies desired, card number 999 in the program listing is used. The number of frequencies may not exceed twice the number of elements used.

```
C-----
C-----INSTRUCTIONS FOR USE OF FINITE ELEMENT PROGRAM
C-----CONTROL CARD
C-----   FIELD I5       NUMBER OF PROBLEMS TO BE WORKED
C-----DATA CARD 1
C-----   FIELD I5       NUMBER OF COORDINATES, N
C-----   FIELD I5       NUMBER OF END CONDITIONS, NEND
C-----   FIELD I5            ROTATORY INERTIA
C-----                            1    NOT INCLUDED
C-----                            2    INCLUDED
C-----   FIELD I5       STRAIGHT SECTION OPTION, KTROL
C-----                       1    THERE ARE CORNERS IN THE STRUCTURE
C-----                       2    THERE ARE NO CORNERS
C-----DATA CARD 2 TO N+1
C-----   FIELD 3F20.0   COORDINATES X, Y, AND NUMBER OF ELEMENTS
C-----                   BETWEEN EACH SET.  LAST CARD, SECTIONS = 0.0
C-----DATA CARD N+2 TO 2N
C-----   FIELD 4F20.0   DENSITY, AREA, MOMENT OF INERTIA ABOUT X AXIS,
C-----                   AND MODULUS OF ELASTICITY FOR EACH SECTION
C-----DATA CARD 2N+1 TO 2N+1+NEND
C-----   FIELD F20.0    ANGLE AT WHICH RIGHT END CONDITIONS ARE APPLIED
C-----   FIELD I5       END CONDITIONS
C-----                       1    CLAMPED - CLAMPED
C-----                       2    CLAMPED - SIMPLE
C-----                       3    CLAMPED - PINNED
C-----                       4    CLAMPED - CLAMPED
C-----                       5    PINNED - FREE
C-----                       6    PINNED - SIMPLE
C-----                       7    PINNED - PINNED
C-----
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION STIFF(100,100), ZM(100,100), ST(5,5), XM(5,5), A(21),
     1          B(21), SEC(21), XIX(20), E(20), AR(20), DEN(20),
     2          EVEC(100,100), EIG(100), EIGEN(4), NU(20), WL(21),
     3          X(100,100), Y(100,100), EIVEC(100,100), FREQ(100),
     4          SECL(21), THETA(21), GAMMA(21), DISPY(4), DISPX(4)
      EQUIVALENCE (X(1,1), EVEC(1,1)), (Y(1,1), ZM(1,1)),
     1           (EIVEC(1,1), STIFF(1,1))
      COMMON ST, XM, C1, C2, C3
      DATA EIGEN /' NATURAL FREQUENCIES(RAD/SEC) '/,
     1     DISPX /' NORMALIZED X DISPLACEMENTS  '/,
     2     DISPY /' NORMALIZED Y DISPLACEMENTS  '/
  999 LIM = 5
      READ (5,90) NDATA
   90 FORMAT (I5)
```

```
      DO 1 IIII=1,NDATA
      READ (5,100) N, NEND, INERT, KTROL
  100 FORMAT (4I5)
      READ (5,101) (A(I), B(I), SEC(I), I=1,N)
  101 FORMAT (3F20.0)
      NN = N - 1
      READ (5,102) (DEN(I), AR(I), XIX(I), E(I), I=1,NN)
  102 FORMAT (4F20.0)
      DO 1 NUM=1,NEND
      READ (5,103) PHI, II
  103 FORMAT (F20.0, I5)
      WRITE (6,104)
  104 FORMAT (1H1, 22H FINITE ELEMENT METHOD)
      WRITE (6,105)
  105 FORMAT (1H0, 8X, 11HX COORD(IN), 9X, 11HY COORD(IN), 7X, 18HNUMBER
     1 OF SECTIONS, 2X, 21HSPECIFIC WT(LB/CU IN), 4X, 11HAREA(SQ IN),
     2 6X, 12HMOMENT(IN 4))
      WRITE (6,107) (A(I), B(I), SEC(I), DEN(I), AR(I), XIX(I), I=1,NN)
  107 FORMAT (2F20.5, /, 40X, 4F20.5)
      WRITE (6,400) A(N), B(N)
  400 FORMAT (2F20.5)
      GO TO (73, 74, 75, 76, 77, 78, 79), II
   73 WRITE (6,110)
  110 FORMAT (1H0, 27H CLAMPED FREE END CONDITION)
      GO TO 80
   74 WRITE (6,111)
  111 FORMAT (1H0, 29H CLAMPED SIMPLE END CONDITION)
      GO TO 80
   75 WRITE (6,112)
  112 FORMAT (1H0, 29H CLAMPED PINNED END CONDITION)
      GO TO 80
   76 WRITE (6,113)
  113 FORMAT (1H0, 30H CLAMPED CLAMPED END CONDITION)
      GO TO 80
   77 WRITE (6,114)
  114 FORMAT (1H0, 26H PINNED FREE END CONDITION)
      GO TO 80
   78 WRITE (6,115)
  115 FORMAT (1H0, 28H PINNED SIMPLE END CONDITION)
      GO TO 80
   79 WRITE (6,116)
  116 FORMAT (1H0, 28H PINNED PINNED END CONDITION)
   80 IF (INERT.NE.1) GO TO 225
      WRITE (6,226)
  226 FORMAT (1H0, 33H ROTATORY INERTIA IS NOT INCLUDED)
```

```
      GO TO 227
  225 WRITE (6,228)
  228 FORMAT (1H0, 29H ROTATORY INERTIA IS INCLUDED)
  227 CONTINUE
C-----
C-----GENERATION AND COMBINATION OF ELEMENTAL STIFFNESS AND MASS MATRICE
C-----
      DO 2 I=1,NN
      WL(I) = DEN(I) * AR(I) / 0.386D3
      XL = DSQRT((A(I+1) -A(I))**2 + (B(I+1)- B(I))**2)
    2 SECL(I) = XL / SEC(I)
      PI = 3.14159265358979
      PI2 = 0.5D0 * PI
      DO 10 J=1,NN
      IF (A(J+1)-A(J)) 11, 12, 11
   12 THETA(J) = 0.5D0 * PI
      IF (B(J+1).LT.B(J)) THETA(J) = - THETA(J)
      GO TO 10
   11 ARG = (B(J+1)-B(J)) / (A(J+1) - A(J))
      THETA(J) = DATAN (ARG)
   10 CONTINUE
      THETA(N) = PHI * PI / 0.180D3
      ACTUAL = 0.0
      DO 13 I=1,N
      GAMMA(I) = THETA(I) - ACTUAL
   13 ACTUAL = ACTUAL + GAMMA(I)
      TOTAL = 0.0
      DO 3 I=1,NN
    3 TOTAL = TOTAL + SEC(I)
      M = TOTAL * 2.0 + 3.0
      DO 4 I=1,M
      DO 4 J=1,M
      STIFF(I,J) = 0.0
    4 ZM(I,J) = 0.0
      IST = 1
      NU(1) = 1
      JJ = 0
      L = SEC(1) - 1
      DO 5 KK=1,NN
      C1 = E(KK) * XIX(KK) / (SECL(KK)**3)
      C2 = WL(KK) * SECL(KK) / 0.420D3
      C3 = WL(KK) * XIX(KK) / (SECL(KK) * AR(KK) * 0.30D2)
      IF (INERT.EQ.1) C3=0.0
      CALL ELEM (SECL(KK))
      IF (KK.EQ.NN) GO TO 62
      IF (DABS(GAMMA(KK+1)).LT.0.1D-5) GO TO 62
      GO TO 63
   62 L = L + 1
   63 IF (L.LT.1) GO TO 64
      DO 7 K=1,L
      DO 6 I=2,5
      DO 6 J=2,5
      STIFF(JJ+I,JJ+J) = STIFF(JJ+I,JJ+J) + ST(I,J)
    6 ZM(JJ+I,JJ+J) = ZM(JJ+I,JJ+J) + XM(I,J)
      JJ = JJ + 2
    7 ZM(IST,IST) = ZM(IST,IST) + XM(1,1)
   64 IF (KK.EQ.NN) GO TO 5
      IF (DABS(GAMMA(KK+1)).LT.0.1D-5) GO TO 5
      ANGLE = GAMMA(KK+1)
C-----
C-----CORNER TRANSFORMATIONS
C-----
      CALL CORNER (ANGLE, 1)
      ZM(IST,IST) = ZM(IST,IST) + XM(1,1)
      STIFF(IST,IST) = STIFF(IST,IST) + ST(1,1)
      DO 8 I=2,5
      ZM(JJ+I,IST) = ZM(JJ+I,IST) + XM(I,1)
      ZM(IST,JJ+I) = ZM(IST,JJ+I) + XM(1,I)
      STIFF(IST,JJ+I) = STIFF(IST,JJ+I) + ST(1,I)
    8 STIFF(JJ+I,IST) = STIFF(JJ+I,IST) + ST(I,1)
      DO 9 I=2,5
      DO 9 J=2,5
      STIFF(JJ+I,JJ+J) = STIFF(JJ+I,JJ+J) + ST(I,J)
    9 ZM(JJ+I,JJ+J) = ZM(JJ+I,JJ+J) + XM(I,J)
      JJ = JJ + 2
      IF (KK.EQ.NN) GO TO 5
      C1 = E(KK+1) * XIX(KK+1) / (SECL(KK+1)**3)
      C2 = WL(KK+1) * SECL(KK+1) / 0.420D3
      C3 = WL(KK+1) * XIX(KK+1) / (SECL(KK+1) * AR(KK+1) * 0.30D2)
      IF (INERT.EQ.1) C3=0.0
      CALL ELEM (SECL(KK+1))
      CALL CORNER (ANGLE,2)
      ZM(IST,IST) = ZM(IST,IST) + XM(1,1)
      STIFF(IST,IST) = STIFF(IST,IST) + ST(1,1)
      DO 15 I=2,5
      ZM(IST,JJ+I) = ZM(IST,JJ+I) + XM(1,I)
      ZM(JJ+I,IST) = ZM(JJ+I,IST) + XM(I,1)
      STIFF(IST,JJ+I) = STIFF(IST,JJ+I) + ST(1,I)
   15 STIFF(JJ+I,IST) = STIFF(JJ+I,IST) + ST(I,1)
      DO 14 I=2,5
      DO 14 J=2,5
      STIFF(JJ+I,JJ+J) = STIFF(JJ+I,JJ+J) + ST(I,J)
   14 ZM(JJ+I,JJ+J) = ZM(JJ+I,JJ+J) + XM(I,J)
      JJ = JJ + 2
      IST = JJ
      NU(KK+1) = IST
      L = SEC(KK+1) - 2
    5 CONTINUE
C-----
C-----APPLICATION OF END CONDITIONS
C-----
      KU = 0
      KR = 0
      GO TO (16,16,16,16,17,17,17,17), II
   16 KL = 3
      GO TO 18
   17 KL = 2
   18 GO TO (19, 20, 21, 22, 19, 20, 21, 22), II
   20 IF (DABS(GAMMA(N)).GT.0.15D1) GO TO 60
      KR = 1
      GO TO 23
   60 KU = IST
      GO TO 61
   21 KR = 1
      KU = IST
      GO TO 23
   22 KR = 2
      KU = IST
   23 MM = M - 2
      DO 25 I=1,KR
      DO 24 J=1,M
      STIFF(MM+I,J) = 0.0
      STIFF(J,MM+I) = 0.0
      ZM(MM+I,J) = 0.0
   24 ZM(J,MM+I) = 0.0
   25 ZM(MM+I,MM+I) = 0.1D1
      IF (KU.EQ.0) GO TO 19
   61 DO 26 K=1,M
      STIFF(KU,K) = 0.0
      STIFF(K,KU) = 0.0
      ZM(KU,K) = 0.0
   26 ZM(K,KU) = 0.0
      ZM(KU,KU) = 0.1D1
   19 DO 28 I=1,KL
      DO 27 J=1,M
      STIFF(I,J) = 0.0
      STIFF(J,I) = 0.0
      ZM(I,J) = 0.0
   27 ZM(J,I) = 0.0
   28 ZM(I,I) = 0.1D1
C-----
C-----MAKING SURE OFF DIAGONAL ELEMENTS IN MASS MATRIX ARE EQUAL
C-----
      DO 29 I=1,M
      K = I + 1
      DO 29 J=K,M
      ZM(I,J) = 0.5D0 * (ZM(I,J) + ZM(J,I))
      ZM(J,I) = ZM(I,J)
   29 CONTINUE
      CALL JACVAT (ZM, M, 1, EIG, EVEC, 100)
C-----
C-----EIGENVALUE PROBLEM ON THE STIFFNESS MATRIX
C-----
      DO 30 I=1,M
      EIG(I) = DABS(EIG(I))
   30 EIG(I) = 0.1D1 / DSQRT(EIG(I))
      DO 31 L=1,M
      DO 31 I=1,M
      ZM(I,L) = 0.0
      DO 31 J=1,M
   31 ZM(I,L) = ZM(I,L) + STIFF(I,J) * EVEC(J,L) * EIG(L)
      DO 32 L=1,M
      DO 32 I=1,M
      STIFF(I,L) = 0.0
      DO 32 J=1,M
   32 STIFF(I,L) = STIFF(I,L) + EIG(I) * EVEC(J,I) * ZM(J,L)
C-----
C-----MAKING SURE OFF DIAGONAL ELEMENTS OF STIFFNESS MATRIX ARE EQUAL
C-----
      DO 33 I=1,M
      K = I + 1
      DO 33 J=K,M
      STIFF(I,J) = 0.5D0 * (STIFF(I,J) + STIFF(J,I))
      STIFF(J,I) = STIFF(I,J)
   33 CONTINUE
      CALL JACVAT (STIFF, M, 1, FREQ, ZM, 100)
C-----
C-----OBTAINING MODE SHAPES FROM EIGENVECTORS
C-----
      DO 49 L=1,M
      DO 49 I=1,M
      EIVEC(I,L) = 0.0
      DO 49 J=1,M
   49 EIVEC(I,L) = EIVEC(I,L) + EVEC(I,J) * EIG(J) * ZM(J,L)
      DO 34 J=1,M
   34 EIG(J) = DSQRT(FREQ(J))
C-----
C-----SIFTING TO PUT IN ORDER OF ASCENDING FREQUENCY
C-----
      MN = M - KR - KL - 1

      IF (LIM.GT.MN) LIM=MN
      LL = LIM + 1
      DO 35 L=1,LL
      TEST = EIG(L)
      NCOL = L
      IL = L + 1
      DO 36 I=IL,M
      IF (TEST.EQ.0.0) GO TO 39
      IF(EIG(I).EQ.0.0) GO TO 36
   37 IF (EIG(I)-TEST) 39, 36, 36
   39 TEST = EIG(I)
      NCOL = I
   36 CONTINUE
      DO 40 J=1,M
      ZM(J,L) = EIVEC(J,NCOL)
      EIVEC(J,NCOL) = EIVEC(J,L)
   40 EIVEC(J,L) = ZM(J,L)
      EIG(NCOL) = EIG(L)
   35 EIG(L) = TEST
      CALL MAPRIN (EIG, 1, LIM, 1, 2, EIGEN)
C-----
C-----X AND Y DISPLACEMENTS
C-----
      DO 41 J=1,LIM
      X(J,1) = 0.0
   41 Y(J,1) = 0.0
      DO 42 K=1,LIM
      JL = 2
      LK = NU(1)
      DO 43 L=1,NN
      JL = JL + SEC(L)
      X(K,2*L) = EIVEC(LK,K)
      Y(K,2*L) = EIVEC(JL,K)
      JL = JL + SEC(L)
      IF (L.EQ.NN) GO TO 44
      IF (DABS(GAMMA(L+1)).LT.0.1D-5) GO TO 44
      X(K,2*L+1) = EIVEC(LK,K)
      LK = NU(L+1)
      IF (DABS(GAMMA(L+1)).NE.PI2) GO TO 45
      IF (GAMMA(L+1)) 46, 47, 47
   47 S = 0.1D1
      T = 0.0
      GO TO 48
   46 S = - 0.1D1
      T = 0.0
      GO TO 48
   45 S = DSIN(GAMMA(L+1))
      T = DCOS(GAMMA(L+1)) / S
   48 Y(K,2*L+1) = -X(K,2*L) * T + EIVEC(LK,K) / S
      GO TO 43
   44 X(K,2*L+1) = X(K,2*L)
      Y(K,2*L+1) = EIVEC(JL,K)
   43 CONTINUE
   42 CONTINUE
      JK = 2 * NN + 1
C-----
C-----PUTTING DISPLACEMENTS IN REAL COORDINATE SYSTEM
C-----
      DO 50 L=2,JK
      DO 50 I=1,LIM
      K = L / 2
      IF (DABS(THETA(K)).NE.PI2) GO TO 55
      S = 0.1D1
      C = 0.0
      IF (THETA(K).LT.0.0) S=-S
      GO TO 56
   55 S = DSIN(THETA(K))
      C = DCOS(THETA(K))
   56 XN = -S * Y(I,L) + C * X(I,L)
      YN = C * Y(I,L) + S * X(I,L)
      X(I,L) = XN
   50 Y(I,L) = YN
C-----
C-----NORMALIZING DISPLACEMENTS
C-----
      DO 52 I=1,LIM
      SUM1 = 0.0
      DO 53 J=1,JK
   53 SUM1 = SUM1 + Y(I,J)**2 + X(I,J)**2
```

```
      SUM = DSQRT(SUM1)
      DO 52 J=1,JK
      X(I,J) = X(I,J) / SUM
   52 Y(I,J) = Y(I,J) / SUM
      CALL MAPRIN (X, LIM, JK, 100, 6, DISPX)
      CALL MAPRIN (Y, LIM, JK, 100, 6, DISPY)
    1 CONTINUE
      STOP
      END




      SUBROUTINE JACVAT(A,N,NOYES,EIVU,EIVR,NDIM)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION A(NDIM,NDIM),EIVU(NDIM),EIVR(NDIM,NDIM)
      IF(N-1)20,23,21
   20 PRINT 22,N
   22 FORMAT(4H N= ,I3,68H IS TOO SMALL.  LIMIT IS 1.  RETURN TO CALLING
     1 ROUTINE FROM JACVAT. )
      RETURN
   23 PRINT 24, A(1,1)
   24 FORMAT (24H IN JACVAT , MATRIX A = ,E14.6)
      RETURN
   21 IF(N-160)1,1,3
    3 PRINT 2,N
    2 FORMAT(3H N=,I5,70H IS TOO LARGE.  LIMIT IS 160.  RETURN TO CALLI-
     1NG ROUTINE FROM JACVAT. )
      RETURN
    1 IF(NOYES)99,102,99
   99 CONTINUE
      DO 101 J=1,N
      DO 100 I=1,N
  100 EIVR(I,J)=0.0
  101 EIVR(J,J)=1.0
  102 ATOP=0.
      DO 112 J=1,N
      DO 111 I=1,J
      IF(A(I,J)-A(J,I))90,103,90
   90 PRINT 106,N,N
  106 FORMAT(14H IN JACVAT (A(,I3,1H,,I3,3H)),)
      PRINT 108,I,J,J,I
```

```
108 FORMAT(3H A(,I3,1H,,I3,10H)  AND  A(,I3,1H,,I3,54H) WERE UNEQUAL,
   1SO THEY WERE REPLACED WITH THEIR MEAN )
    A(I,J)=.5*(A(I,J)+A(J,I))
    A(J,I)=A(I,J)
103 CONTINUE
    IF(ATOP-DABS(A(I,J)))104,111,111
104 ATOP=DABS(A(I,J))
111 CONTINUE
112 EIVU(J)=A(J,J)
    IF(ATOP)109,109,113
109 PRINT 110
110 FORMAT(26H IN JACVAT, MATRIX A  = C )
    RETURN
113 AVGF=DFLOAT(N*(N-1))*.55
    D=0.0
    DO 114  JJ=2,N
    DO 114  II=2,JJ
    S=A(II-1,JJ)/ATOP
114 D=S*S+D
    DSTOP=(1.E-06)*D
    THRSH = DSQRT(C/AVGF)*ATOP
115 IFLAG=0
    DO 130 JCOL=2,N
    JCOL1=JCOL-1
    DO 130 IROW=1,JCOL1
    AIJ=A(IROW,JCOL)
    IF(DABS(AIJ)-THRSH)130,130,117
117 AII=A(IROW,IROW)
    AJJ=A(JCOL,JCOL)
    S=AJJ-AII
    IF(DABS(AIJ)-1.E-12*DABS(S))130,130,118
118 IFLAG=1
    IF(1.E-10*DABS(AIJ)-DABS(S))116,119,119
119 S=.707106781
    C=S
    GO TO 120
116 T=AIJ/S
    S=0.25/DSQRT(0.25+T*T)
    C=DSQRT(0.5+S)
    S=2.*T*S/C
120 DO 121  I=1,IROW
    T=A(I,IROW)
    U=A(I,JCOL)
    A(I,IROW)=C*T-S*U
121 A(I,JCOL)=S*T+C*U
```

```
      I2=IROW+2
      IF(I2-JCOL)127,127,123
  127 CONTINUE
      DO 122  I=I2,JCOL
      T=A(I-1,JCOL)
      U=A(IROW,I-1)
      A(I-1,JCOL)=S*U+C*T
  122 A(IROW,I-1)=C*U-S*T
  123 A(JCOL,JCOL)=S*AIJ+C*AJJ
      A(IROW,IROW)=C*A(IROW,IROW)-S*(C*AIJ-S*AJJ)
      DO 124  J=JCOL,N
      T=A(IROW,J)
      U=A(JCOL,J)
      A(IROW,J)=C*T-S*U
  124 A(JCOL,J)=S*T+C*U
      IF(NOYES)131,126,131
  131 CONTINUE
      DO 125  I=1,N
      T=EIVR(I,IROW)
      EIVR(I,IROW)=C*T-EIVR(I,JCOL)*S
  125 EIVR(I,JCOL)=S*T+EIVR(I,JCOL)*C
  126 CONTINUE
      S=AIJ/ATOP
      D=D-S*S
      IF(D-DSTOP)1260,129,129
 1260 D=0.
      DO 128  JJ=2,N
      DO 128  II=2,JJ
      S=A(II-1,JJ)/ATOP
  128 D=S*S+D
      DSTOP=(1.E-06)*D
  129 THRSH=DSQRT(D/AVGF)*ATOP
  130 CONTINUE
      IF(IFLAG)115,134,115
  134 T=A(1,1)
      A(1,1)=EIVU(1)
      EIVU(1)=T
      DO 132  J=2,N
      T=A(J,J)
      A(J,J)=EIVU(J)
      EIVU(J)=T
      DO 132  I=2,J
  132 A(I-1,J)=A(J,I-1)
  133 RETURN
      END
      SUBROUTINE ELEM (XL)
C-----
C-----SUBROUTINE GENERATES ELEMENTAL STIFFNESS AND MASS MATRICES
C-----
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION ST(5,5), XM(5,5)
      COMMON ST, XM, C1, C2, C3
      DO 1 I=1,5
      ST(1,I) = 0.0
      ST(I,1) = 0.0
      XM(1,I) = 0.0
    1 XM(I,1) = 0.0
      ST(2,2) = C1 * 0.12D2
      ST(2,3) = C1 * 0.60D1 * XL
      ST(2,4) = - ST(2,2)
      ST(2,5) = ST(2,3)
      ST(3,3) = C1 * 0.4D1 * XL**2
      ST(3,4) = - ST(2,3)
      ST(3,5) = C1 * 0.2D1 * XL**2
      ST(4,4) = ST(2,2)
      ST(4,5) = - ST(2,3)
      ST(5,5) = ST(3,3)
      XM(1,1) = C2 * 0.420D3
      XM(2,2) = C3 * 0.36D2 + C2 * 0.156D3
      XM(2,3) = (C3 * 0.30D1 + C2 * 0.22D2) * XL
      XM(2,4) =-C3 * 0.36D2 + C2 * 0.54D2
      XM(2,5) = (C3 * 0.30D1 - C2 * 0.13D2) * XL
      XM(3,3) = (C3 * 0.40D1 + C2 * 0.40D1) * XL**2
      XM(3,4) = (-C3 * 0.30D1 + C2 * 0.13D2) * XL
      XM(3,5) = -(C3 + C2 * 0.30D1) * XL**2
      XM(4,4) = C3 * 0.36D2 + C2 * 0.156D3
      XM(4,5) = -(C3 * 0.3D1 + C2 * 0.22D2) * XL
      XM(5,5) = (C3 * 0.40D1 + C2 * 0.40D1) * XL**2
      DO 2 I=2,5
      DO 2 J=I,5
      ST(J,I) = ST(I,J)
    2 XM(J,I) = XM(I,J)
      RETURN
      END
      SUBROUTINE MAPRIN (A,N,M,ND,ICODE,TITLE)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION A(ND,1), TITLE(4), NKOL(6), F(4,6), FM1(2), FM2(2),
     1 F1(4), F2(4), F3(4), F4(4), F5(4), F6(4)
      EQUIVALENCE (F1,F),(F2,F(5)),(F3,F(9)),(F4,F(13)),(F5,F(17)),
     1 (F6,F(21))
C
C     MAPRIN PRINTS 'N' ROWS AND 'M' COLUMNS OF MATRIX A (ROW
C     DIMENSION = ND) BY PARTITIONING INTO GROUPS OF 'NK' COLUMNS.
C     ICODE = 1 TO 6 SPECIFIES THE FORMAT (SEE DATA STATEMENT FOR F).
C     THIS VERSION FOR IBM 360.
C
      DATA NKOL /8,8,10,10,12,12/
      DATA   F1 /32H(/2X 8I15)       (I5,2X, 8D15.7)/,
     1       F2 /32H(/2X 8I15)       (I5,1X, 8F15.3)/,
     2       F3 /32H(/3X10I12)       (I5,2X,10D12.4)/,
     3       F4 /32H(/3X10I12)       (I5,1X,10F12.6)/,
     4       F5 /32H(/4X12I10)       (I5,2X,12D10.2)/,
     5       F6 /32H(/4X12I10)       (I5,1X,12F10.4)/
      IF (ICODE.LT.1.OR.ICODE.GT.6)  ICODE = 1
      NK = NKOL(ICODE)
      FM1(1) = F(1,ICODE)
      FM1(2) = F(2,ICODE)
      FM2(1) = F(3,ICODE)
      FM2(2) = F(4,ICODE)
      NPR = 1 + (M-1)/NK
      N1 = 1
      N2 = MINO(M,NK)
      PRINT 10, TITLE
   10 FORMAT (/1H0,4A8)
      DO 150  K = 1,NPR
      PRINT FM1, (J, J=N1,N2)
      DO 140  I = 1,N
  140 PRINT FM2, I, (A(I,J), J=N1,N2)
      N1 = N1 + NK
  150 N2 = MINO(M,N2+NK)
      RETURN
      END
      SUBROUTINE CORNER (ANGLE,NSIDE)
C-----
C-----SUBROUTINE PERFORMS CORNER TRANSFORMATION
C-----   NSIDE=1, LAST ELEMENT BEFORE CORNER IS TRANSFORMED
C-----   NSIDE=2, FIRST ELEMENT AFTER CORNER IS TRANSFORMED
C-----
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION ST(5,5), XM(5,5), ROTA1(5,5), ROTA2(5,5), ROTA3(5,5)
      COMMON ST, XM, C1, C2, C3
      PI = 0.31415926536D1
      PI2 = 0.5D0 * PI
      DO 1 I=1,5
      DO 2 J=1,5
      ROTA1(I,J) = 0.0
    2 ROTA2(I,J) = 0.0
      ROTA1(I,I) = 0.1D1
    1 ROTA2(I,I) = 0.1D1
      IF (DABS(ANGLE).NE.PI2) GO TO 8
      IF (ANGLE) 9,10,10
   10 S = 0.1D1
      T = 0.0
      GO TO 11
    9 S = -0.1D1
      T = 0.0
      GO TO 11
    8 S = DSIN(ANGLE)
      T = DCOS(ANGLE) / S
   11 ROTA1(4,1) = - T
      ROTA1(4,4) = ROTA1(4,4) / S
      ROTA2(1,1) = 0.0
      ROTA2(1,2) = 0.1D1
      ROTA2(2,1) =-0.1D1 / S
      ROTA2(2,2) = T
      IF (NSIDE.EQ.2) GO TO 5
      DO 3 L=1,5
      DO 3 I=1,5
      ROTA2(I,L) = 0.0
      ROTA3(I,L) = 0.0
```

```
  DO 3 J=1,5
  ROTA2(I,L) = ROTA2(I,L) + ST(I,J) * ROTA1(J,L)
3 ROTA3(I,L) = ROTA3(I,L) + XM(I,J) * ROTA1(J,L)
  DO 4 L=1,5
  DO 4 I=1,5
  ST(I,L) = 0.0
  XM(I,L) = 0.0
  DO 4 J=1,5
  ST(I,L) = ST(I,L) + ROTA1(J,I) * ROTA2(J,L)
4 XM(I,L) = XM(I,L) + ROTA1(J,I) * ROTA3(J,L)
  RETURN
5 DO 6 L=1,5
  DO 6 I=1,5
  ROTA1(I,L) = 0.0
  ROTA3(I,L) = 0.0
  DO 6 J=1,5
  ROTA1(I,L) = ROTA1(I,L) + ST(I,J) * ROTA2(J,L)
6 ROTA3(I,L) = ROTA3(I,L) + XM(I,J) * ROTA2(J,L)
  DO 7 L=1,5
  DO 7 I=1,5
  ST(I,L) = 0.0
  XM(I,L) = 0.0
  DO 7 J=1,5
  ST(I,L) = ST(I,L) + ROTA2(J,I) * ROTA1(J,L)
7 XM(I,L) = XM(I,L) + ROTA2(J,I) * ROTA3(J,L)
  RETURN
  END
```

```
FINITE ELEMENT METHOO
       X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(L8/CU IN)       AREA(SO IN)           MOMENT(IN 4)
          0.0                  0.0
                                                      2.00000             0.28300               4.62000               30.30000
          0.0               120.00000
                                                      4.00000             0.28300              29.40000              625.00000
       360.00000            192.00000
                                                      4.00000             0.28300              29.40000              625.00000
       720.00000            120.00000
                                                      2.00000             0.28300               5.88000               82.50000
       720.00000              0.0
CLAMPEO CLAMPEO ENO CONOITION
ROTATORY INERTIA IS NOT INCLUOEO

NATURAL FREQUENCIES(RAO/SEC)
                1           2              3             4              5
  1          25.752      34.679         79.647       150.444         280.597

NORMALIZEO X OISPLACEMENTS
               1          2          3          4          5          6          7          8          9
  1      0.0        0.2051     0.3680     0.3016     0.2578     0.2313     0.1476     0.0365     0.0
  2      0.0        0.0932     0.2685     0.3468     0.3440     0.3988     0.4196     0.1980     0.0
  3      0.0        0.1615     0.1677     0.0421     0.1812     0.0612     0.1946     0.1604     0.0
  4      0.0        0.0551    -0.1330    -0.2284    -0.0126     0.2130     0.1077    -0.0587     0.0
  5      0.0       -0.7539    -0.0475    -0.0431    -0.0348     0.0153    -0.0222    -0.6213     0.0

NORMALIZEO Y OISPLACEMENTS
               1          2          3          4          5          6          7          8          9
  1      0.0        0.0        0.0        0.3317     0.5509     0.4183     0.0000     0.0        0.0
  2      0.0        0.0        0.0       -0.3912    -0.3775    -0.1036     0.0000     0.0        0.0
  3      0.0        0.0        0.0        0.6279    -0.0674    -0.6671     0.0000     0.0        0.0
  4      0.0        0.0        0.0        0.4771    -0.6016     0.5263     0.0000     0.0        0.0
  5      0.0        0.0        0.0       -0.0217    -0.0632     0.1872    -0.0000     0.0        0.0
```

```
FINITE ELEMENT METHOO
       X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(L8/CU IN)       AREA(SO IN)           MOMENT(IN 4)
          0.0                  0.0
                                                      2.00000             0.28300               4.62000               30.30000
          0.0               120.00000
                                                      4.00000             0.28300              29.40000              625.00000
       360.00000            192.00000
                                                      4.00000             0.28300              29.40000              625.00000
       720.00000            120.00000
                                                      2.00000             0.28300               5.88000               82.50000
       720.00000              0.0
PINNEO PINNEO ENO CONOITION
ROTATORY INERTIA IS NOT INCLUOEO

NATURAL FREOUENCIES(RAO/SEC)
                1           2              3             4              5
  1          15.256      25.364         75.667       149.820         278.160

NORMALIZEO X OISPLACEMENTS
               1          2          3          4          5          6          7          8          9
  1      0.0        0.2982     0.4299     0.4152     0.3960     0.3957     0.3622     0.2170     0.0
  2      0.0        0.0689     0.0054    -0.0927    -0.1257    -0.1777    -0.2569    -0.2132     0.0
  3      0.0       -0.2040    -0.1144     0.0154    -0.1232     0.0021    -0.1319    -0.1993     0.0
  4      0.0       -0.0993     0.1289     0.2238     0.0078    -0.2159    -0.1133     0.0961     0.0
  5      0.0       -0.7862    -0.0051    -0.0060     0.0013     0.0229     0.0078    -0.6119     0.0

NORMALIZEO Y OISPLACEMENTS
               1          2          3          4          5          6          7          8          9
  1      0.0        0.0        0.0        0.0733     0.1693     0.1678     0.0000     0.0        0.0
  2      0.0        0.0        0.0        0.4906     0.6559     0.3961    -0.0000     0.0        0.0
  3      0.0        0.0        0.0       -0.6493     0.0438     0.6702    -0.0000     0.0        0.0
  4      0.0        0.0        0.0       -0.4743     0.6054    -0.5130    -0.0000     0.0        0.0
  5      0.0        0.0        0.0        0.0046    -0.0322     0.0756     0.0000     0.0        0.0
```

```
FINITE ELEMENT METHOD
      X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)    AREA(SQ IN)       MOMENT(IN 4)
          0.0                  0.0
                                                     2.00000             0.28300             5.88000          82.50000
          0.0              120.00000
                                                     2.00000             0.28300            29.40000         625.00000
      360.00000            192.00000
                                                     2.00000             0.28300            29.40000         625.00000
      720.00000            120.00000
                                                     2.00000             0.28300             4.62000          30.30000
      720.00000              0.0
CLAMPED CLAMPED END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)

               1              2                3               4               5
  1          25.761         34.694           79.974         152.647         311.542

NORMALIZED X DISPLACEMENTS

             1          2         3          4          5          6          7          8          9
  1        0.0      -0.0366   -0.1478   -0.2315    -0.2580    -0.3018    -0.3681    -0.2051     0.0
  2        0.0       0.1980    0.4194    0.3985     0.3437     0.3463     0.2680     0.0929     0.0
  3        0.0      -0.1596   -0.1928   -0.0592    -0.1793    -0.0401    -0.1659    -0.1608     0.0
  4        0.0       0.0576   -0.1079   -0.2134     0.0121     0.2278     0.1321    -0.0545     0.0
  5        0.0      -0.6113   -0.0122    0.0212    -0.0237    -0.0320    -0.0351    -0.7690     0.0

NORMALIZED Y DISPLACEMENTS

             1          2         3          4          5          6          7          8          9
  1        0.0        0.0       0.0      0.4182     0.5507     0.3315    -0.0000     0.0        0.0
  2        0.0        0.0       0.0      0.1044     0.3785     0.3916     0.0000     0.0        0.0
  3        0.0        0.0       0.0     -0.6682    -0.0673     0.6291    -0.0000     0.0        0.0
  4        0.0        0.0       0.0      0.5272    -0.6001     0.4786     0.0000     0.0        0.0
  5        0.0        0.0       0.0     -0.1670     0.0573     0.0157    -0.0000     0.0        0.0
```

```
FINITE ELEMENT METHOD
      X COORD(IN)          Y COORD(IN)          NUMBER OF SECTIONS  SPECIFIC WT(LB/CU IN)    AREA(SQ IN)       MOMENT(IN 4)
          0.0                  0.0
                                                     4.00000             0.28300             5.88000          82.50000
          0.0              120.00000
                                                     4.00000             0.28300            29.40000         625.00000
      360.00000            192.00000
                                                     4.00000             0.28300            29.40000         625.00000
      720.00000            120.00000
                                                     4.00000             0.28300             4.62000          30.30000
      720.00000              0.0
CLAMPED CLAMPED END CONDITION
ROTATORY INERTIA IS NOT INCLUDED

NATURAL FREQUENCIES(RAD/SEC)

               1              2                3               4               5
  1          25.752         34.679           79.647         150.444         280.594

NORMALIZED X DISPLACEMENTS

             1          2         3          4          5          6          7          8          9
  1        0.0      -0.0365   -0.1476   -0.2313    -0.2578    -0.3016    -0.3680    -0.2051     0.0
  2        0.0       0.1980    0.4196    0.3988     0.3440     0.3468     0.2685     0.0932     0.0
  3        0.0      -0.1604   -0.1946   -0.0612    -0.1812    -0.0421    -0.1677    -0.1615     0.0
  4        0.0       0.0587   -0.1077   -0.2130     0.0126     0.2284     0.1330    -0.0552     0.0
  5        0.0       0.6211    0.0221   -0.0153     0.0347     0.0430     0.0474     0.7541     0.0

NORMALIZED Y DISPLACEMENTS

             1          2         3          4          5          6          7          8          9
  1        0.0        0.0       0.0      0.4183     0.5509     0.3317    -0.0000     0.0        0.0
  2        0.0        0.0       0.0      0.1036     0.3775     0.3912     0.0000     0.0        0.0
  3        0.0        0.0       0.0     -0.6671    -0.0674     0.6279    -0.0000     0.0        0.0
  4        0.0        0.0       0.0      0.5263    -0.6016     0.4771     0.0000     0.0        0.0
  5        0.0        0.0       0.0      0.1871    -0.0632    -0.0217     0.0000     0.0        0.0
```

APPENDIX D

TRANSFER MATRIX PROGRAM

General

Program DYNRES is a double precision IBM FORTRAN IV language[7] digital computer program designed to obtain "exact" solutions for the principal modes of straight sided plane frames of uniform cross section. These modes are used as a standard for comparing the LUMPED and FINITE programs. Axial and shear deformation, as well as rotatory inertia, are neglected.

The program uses the method of false position[9] to assist convergence to the frequencies. Convergence is obtained when either the absolute value of the frequency determinant is less than 10^{-16}, or the frequency is bracketed such that the interval is less than 10^{-9}.

The program is dimensioned for use with structures having no more than nine members. The program requires a total storage of 50K bytes. Each member must have uniform properties, but the properties may vary from member to member.

The more severely constrained end of the structure is designated the left end. This end must be either fixed or pinned. The right end is allowed any of the four common boundary conditions: fixed, pinned, simply supported, or free. The structure may not have any intermediate supports or externally applied forces.

Program Structure

The program consists of the main body and two subroutines. The program requires the use of double precision sine, cosine, arctangent, hyperbolic sine, hyperbolic cosine, and square root routines. If these routines are not available in the FORTRAN library, they must be supplied by other means. The main body is prefaced with a comment section containing the format and makeup of the input deck. The functions of the various sections of the main program are explained in comment cards spaced throughout it.

The subroutine EVAL generates the transfer matrices and evaluates the frequency determinant after a trial frequency has been determined by the main program.

The subroutine MAPRIN is the output subroutine. It performs in the same manner as discussed in the LUMPED program on page 58 of Appendix B.

Input

The input required by the DYNRES program is basically the same as for the LUMPED program (page 58 of Appendix B). The DYNRES program requires the coordinates of the midpoint of each member as well as the end and corner coordinates. No further subdivision of the frame is required. The rest of the data is the same. Format and order of data cards are given at the front of the main program.

Output

The coordinates of ends, midpoints, and corners are automatically printed by the program. The properties (modulus of elasticity, specific weight, area, moment of inertia) between each set of coordinates are also printed. The rest of the output is identical with that of the LUMPED program. The number of frequencies solved for is specified by a control card. Its makeup is shown in the comment section preceding the program.

```
C-----
C-----INSTRUCTIONS FOR USE OF TRANSFER MATRICES PROGRAM
C-----CONTROL CARD
C-----    FIELD I5        NUMBER OF PROBLEMS TO BE WORKED
C-----DATA CARD 1
C-----    FIELD I5        NUMBER OF COORDINATES, AND MIDPOINTS, N
C-----    FIELD I5        NUMBER OF FREQUENCIES DESIRED, LIM
C-----    FIELD I5        NUMBER OF END CONDITIONS, NEND
C-----    FIELD I5        STRAIGHT SECTION OPTION, KTROL
C-----                             1    THERE ARE CORNERS IN THE STRUCTURE
C-----                             2    THERE ARE NO CORNERS
C-----DATA CARD 2 TO N+1
C-----    FIELD 2F20.0    COORDINATES X,Y AND MIDPOINT COORDINATES
C-----DATA CARD N+2 TO 2N
C-----    FIELD 4F20.0    DENSITY, AREA, MOMENT OF INERTIA ABOUT X AXIS,
C-----                     AND MODULUS OF ELASTICITY FOR EACH SECTION
C-----DATA CARD 2N+1 TO 2N+1+NEND
C-----    FIELD F20.0     ANGLE AT WHICH RIGHT END CONDITIONS ARE APPLIED
C-----    FIELD I5        END CONDITIONS
C-----                             1    CLAMPED - CLAMPED
C-----                             2    CLAMPED - SIMPLE
C-----                             3    CLAMPED - PINNED
C-----                             4    CLAMPED - CLAMPED
C-----                             5    PINNED - FREE
C-----                             6    PINNED - SIMPLE
C-----                             7    PINNED - PINNED
C-----
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION U(6,108), SECL(20), A(20), B(20), THETA(20), GAMMA(20),
     1          YDISP(4), DEN(20), XIX(20), AREA(20), WL(20), E(20),
     2          VECB(6,3), FREQ(4),  OMEGA1(10), OMEGA2(10), DET1(10),
     3          DET2(10), ROOT(10), X(10,20), Y(10,20), XDISP(4),
     4          G(6,108), VEC(6,6), VECA(6,3)
      EQUIVALENCE (VEC(1,1), VECA(1,1)), (VEC(1,4), VECB(1,1))
      COMMON U, SECL, VEC, G, XIX, E, WL, KTROL
      DATA FREQ /' NATURAL FREQUENCIES(RAD/SEC) '/,
     1     XDISP /' NORMALIZED X DISPLACEMENTS  '/,
     2     YDISP /' NORMALIZED Y DISPLACEMENTS  '/
      READ (5,299) NDATA
  299 FORMAT (I5)
      DO 99 IIII=1,NDATA
      READ (5,300) N, LIM, NEND, KTROL
  300 FORMAT (4I5)
      READ (5,301) (A(I), B(I), I=1,N)
  301 FORMAT (2F20.0)
```

```
      NN = N - 1
      READ (5,302) (DEN(I), AREA(I), XIX(I), F(I), I=1,NN)
  302 FORMAT (4F20.0)
      DO 99 NENDS=1,NEND
      READ (5,303) PHI, II
  303 FORMAT (F20.0, I5)
      ADD = 0.30D1
      DO 1  I=1,NN
      SECL(I) = DSQRT ((A(I+1) - A(I))**2 + (B(I+1) - B(I))**2)
    1 WL(I) = DEN(I) * AREA(I) / 0.38607D3
      PI = 3.14159265358979
C-----
C     AXIS ROTATION GENERATION SECTION
C-----
      DO 2  J=1,NN
      IF (A(J+1) - A(J)) 101, 100, 101
  100 THETA(J) = 0.5D0 * PI
      IF (B(J+1).LT.B(J)) THETA(J) = - THETA(J)
      GO TO 2
  101 ARG = (B(J+1) - B(J)) / (A(J+1) - A(J))
      THETA(J) = DATAN (ARG)
    2 CONTINUE
      THETA(N) = PHI * PI / 0.180D3
      ACTUAL = 0.0D0
      DO 3 I=1,N
      GAMMA(I) = THETA(I) - ACTUAL
    3 ACTUAL = ACTUAL + GAMMA(I)
      NJ = N * 6
      JJ = 6 * NN
      DO 4  I=1,6
      DO 4  J=1,NJ
    4 G(I,J) = 0.0
      JN = 1
      DO 5  I=1,N
      S = DSIN (GAMMA(I))
      CC = DCOS (GAMMA(I))
      IF (DABS(CC).LT.0.2D-14) CC = 0.0
      IF (KTROL.EQ.1) GO TO 98
      CC = 0.1D1
      S = 0.0
   98 G(1,JN) = CC
      G(2,JN) = - S
      G(1,JN+1) = S
      G(2,JN+1) = CC
      G(3,JN+2) = 0.1D1
      G(4,JN+3) = 0.1D1
      G(5,JN+4) = CC
      G(6,JN+4) = S
      G(5,JN+5) = - S
      G(6,JN+5) = CC
    5 JN = JN + 6
C     CALL MAPRIN (G, 6, NJ, 6, 6, ROTA)
      OMEGA = 0.100D1
C-----
C-----SEARCH FOR ROOT LOCATIONS
C-----
      JR = 1
      JB = 1
      DO 26 L=1,LIM
   21 CALL EVAL (OMEGA, N, II, DET)
  400 FORMAT (1H0, 2D20.10)
      IF (OMEGA.EQ.0.10D2) ADD=0.10D2
      IF (OMEGA.EQ.0.200D3) ADD = 0.50D2
      IF (OMEGA.EQ.0.1000D4) ADD = 0.100D3
      IF (OMEGA.EQ.0.1000D5) ADD = 0.250D3
      IF (DET.LT.0.0) NEW = 1
      IF (DET.GT.0.0) NEW = 2
      IF (DET.EQ.0.0) GO TO 24
      IF (OMEGA.EQ.0.100D1) OLD=NEW
      IF(NEW.NE.OLD) GO TO 22
   23 OLD = NEW
      FORMER = DET
      START = OMEGA
      OMEGA = OMEGA + ADD
      GO TO 21
   24 START = OMEGA
      FORMER = DET
   22 OMEGA1(JB) = START
      OMEGA2(JB) = OMEGA
      DET1(JB) = FORMER
      DET2(JB) = DET
      JB = JB + 1
      GO TO 25
   25 OLD = NEW
      FORMER = DET
      START = OMEGA
   26 OMEGA = OMEGA + ADD
  425 FORMAT (4D20.10)
```

```
C-----
C-----CONVERGENCE TO ROOTS USING A FALSE POSITION ROUTINE
C-----
      DO 40 I=1,LIM
      IF (DET1(I)) 29, 39, 28
   28 OLD = 2
   42 VALUE = OMEGA1(I) + DET1(I) / (DET1(I) - DET2(I)) * (OMEGA2(I) -
     1 OMEGA1(I))
      CALL EVAL (VALUE, N, II, DET)
  401 FORMAT (1HO, 40X, F20.10, D20.10)
      IF (DABS(DET).LT.0.1D-20) GO TO 27
      IF (DET.LT.0.0) NEW = 1
      IF (DET.GT.0.0) NEW = 2
      IF (OLD.NE.NEW) GO TO 38
   36 IF (VALUE.LT.OMEGA1(I)) GO TO 33
      IF (VALUE.GT.OMEGA2(I)) GO TO 33
      OMEGA1(I) = VALUE
      DET1(I) = DET
      IF ((OMEGA2(I) - OMEGA1(I)).LE.0.1D-7) GO TO 27
   30 VALUE = OMEGA1(I) + DET1(I) / (DET1(I) - DET2(I)) * (OMEGA2(I) -
     1 OMEGA1(I))
      CALL EVAL (VALUE, N, II, DET)
      IF (DABS(DET).LT.0.1D-20) GO TO 27
      IF (DET.LT.0.0) NEW = 1
      IF (DET.GT.0.0) NEW = 2
      IF (OLD.NE.NEW) GO TO 38
   32 IF (VALUE.LT.OMEGA1(I)) GO TO 33
      IF (VALUE.GT.OMEGA2(I)) GO TO 33
      START = OMEGA1(I)
      FORMER = DET1(I)
      OMEGA1(I) = VALUE
      DET1(I) = DET
      IF ((OMEGA2(I) - OMEGA1(I)).LE.0.1D-7) GO TO 27
      IF (DABS(DET1(I) - FORMER).LT.0.1D-9) GO TO 42
      VALUE = OMEGA1(I) - DET1(I) / (DET1(I) - FORMER) * (OMEGA1(I) -
     1 START)
      CALL EVAL (VALUE, N, II, DET)
      IF (DABS(DET).LT.0.1D-20) GO TO 27
      IF (DET.LT.0.0) NEW = 1
      IF (DET.GT.0.0) NEW = 2
      IF (OLD.EQ.NEW) GO TO 35
   34 IF (VALUE.GT.OMEGA2(I)) GO TO 42
      IF (VALUE.LT.OMEGA1(I)) GO TO 42
      OMEGA2(I) = VALUE
      DET2(I) = DET
      IF ((OMEGA2(I) - OMEGA1(I)).LE.0.1D-7) GO TO 27
      GO TO 42
   29 OLD = 1
   41 VALUE = OMEGA1(I) + DET1(I) / (DET1(I) - DET2(I)) * (OMEGA2(I) -
     1 OMEGA1(I))
      CALL EVAL (VALUE, N, II, DET)
      IF (DABS(DET).LT.0.1D-20) GO TO 27
      IF (DET.LT.0.0) NEW = 1
      IF (DET.GT.0.0) NEW = 2
      IF (NEW.EQ.OLD) GO TO 36
   38 IF (VALUE.GT.OMEGA2(I)) GO TO 33
      IF (VALUE.LT.OMEGA1(I)) GO TO 33
      OMEGA2(I) = VALUE
      DET2(I) = DET
      IF ((OMEGA2(I) - OMEGA1(I)).LE.0.1D-7) GO TO 27
   37 VALUE = OMEGA1(I) + DET1(I) / (DET1(I) - DET2(I)) * (OMEGA2(I) -
     1 OMEGA1(I))
      CALL EVAL (VALUE, N, II, DET)
      IF (DABS(DET).LT.0.1D-20) GO TO 27
      IF (DET.LT.0.0) NEW = 1
      IF (DET.GT.0.0) NEW = 2
      IF (NEW.EQ.OLD) GO TO 36
   31 IF (VALUE.GT.OMEGA2(I)) GO TO 33
      IF (VALUE.LT.OMEGA1(I)) GO TO 33
      FINISH = OMEGA2(I)
      FORMER = DET2(I)
      OMEGA2(I) = VALUE
      DET2(I) = DET
      IF ((OMEGA2(I) - OMEGA1(I)).LE.0.1D-7) GO TO 27
      IF (DABS(DET2(I) - FORMER).LT.0.1D-9) GO TO 41
      VALUE = OMEGA2(I) + DET2(I) / (DET2(I) - FORMER) * (FINISH -
     1 OMEGA2(I))
      CALL EVAL (VALUE, N, II, DET)
      IF (DABS(DET).LT.0.1D-20) GO TO 27
      IF (DET.LT.0.0) NEW = 1
      IF (DET.GT.0.0) NEW = 2
      IF (OLD.NE.NEW) GO TO 34
   35 IF (VALUE.LT.OMEGA1(I)) GO TO 41
      IF (VALUE.GT.OMEGA2(I)) GO TO 41
      OMEGA1(I) = VALUE
      DET1(I) = DET
      IF ((OMEGA2(I) - OMEGA1(I)).LE.0.1D-7) GO TO 27
      GO TO 41
   39 VALUE = OMEGA1(I)
      GO TO 27
   33 WRITE (6,402) I
  402 FORMAT (1H0, 25HWILL NOT CONVERGE TO ROOT, I3)
      GO TO 27
   27 ROOT(I) = VALUE
C-----
C     X AND Y DISPLACEMENTS
C-----
C     CALL MAPRIN (U, 6, JJ, 6, 6, TRANS)
C     CALL MAPRIN (VEC, 6, 6, 6, 6, ENCON)
      V2 = 0.1D1
      IF (KTROL.GT.1) GO TO 62
   70 DET = VEC(1,1) * VEC(2,3) - VEC(2,1) * VEC(1,3)
      IF (DET.EQ.0.0) GO TO 43
      V1 = (-VEC(1,2) * VEC(2,3) + VEC(2,2) * VEC(1,3)) / DET
      V3 = (-VEC(1,1) * VEC(2,2) + VEC(2,1) * VEC(1,2)) / DET
      GO TO 45
   43 DET = VEC(2,1) * VEC(3,3) - VEC(3,1) * VEC(2,3)
      IF (DET.EQ.0.0) GO TO 44
      V1 = (-VEC(2,2) * VEC(3,3) + VEC(3,2) * VEC(2,3)) / DET
      V3 = (-VEC(2,1) * VEC(3,2) + VEC(3,1) * VEC(2,2)) / DET
      GO TO 45
   44 WRITE (6,404) I
  404 FORMAT (1H0, 31HNO UNIQUE SOLUTION FOR EQUATION, I3)
      GO TO 40
   62 GO TO (63, 63, 60, 60, 63, 63, 60), II
   63 IF (VEC(1,2).EQ.0.0) GO TO 60
      V1 = - VEC(1,2) / VEC(1,1)
      V3 = 0.0
      GO TO 45
   60 IF (VEC(2,2).EQ.0.0) GO TO 61
      V1 = - VEC(2,2) / VEC(2,1)
      V3 = 0.0
      GO TO 45
   61 IF (VEC(3,2).EQ.0.0) GO TO 44
      V1 = - VEC(3,2) / VEC(3,1)
      V3 = 0.0
   45 GO TO (46, 46, 46, 46, 47, 47, 47), II
   46 VEC(3,1) = 0.0
      VEC(4,1) = V1
      GO TO 48
   47 VEC(3,1) = V1
      VEC(4,1) = 0.0
   48 VEC(1,1) = 0.0
      VEC(2,1) = 0.0
      VEC(5,1) = V2
      VEC(6,1) = V3
  405 FORMAT (1H0, 6D15.6)
      X(I,1) = VEC(1,1)
      Y(I,1) = VEC(2,1)
      JN = 0
      DO 51 L=1,NN
      DO 49 K=1,6
      VEC(K,2) = 0.0
      DO 49 J=1,6
   49 VEC(K,2) = VEC(K,2) + G(K,JN+J) * VEC(J,1)
      DO 50 K=1,6
      VEC(K,1) = 0.0
      DO 50 J=1,6
   50 VEC(K,1) = VEC(K,1) + U(K,JN+J) * VEC(J,2)
      X(I,L+1) = -VEC(2,1) * DSIN(THETA(L)) + VEC(1,1)* DCOS(THETA(L))
      Y(I,L+1) = VEC(2,1) * DCOS(THETA(L)) + VEC(1,1) * DSIN(THETA(L))
   51 JN = JN + 6
   40 CONTINUE
C-----
C     LABELING END CONDITIONS
C-----
      WRITE (6,304)
  304 FORMAT (1H1, 23H TRANSFER MATRIX METHOD)
      WRITE (6,407)
  407 FORMAT (1H0, 8X, 11HX COORD(IN), 8X, 11HY COORD(IN), 2X, 26HMODULU
     1S OF ELASTICITY(PSI), 2X, 21HSPECIFIC WT(LB/CU IN), 2X, 11HAREA(SQ
     2 IN), 6X, 12HMOMENT(IN 4))
      WRITE (6,408) (A(I), B(I), E(I), DEN(I), AREA(I), XIX(I), I=1,NN)
  408 FORMAT (2F20.5, /, 40X, 4F20.5)
      WRITE (6,409)  A(N), B(N)
  409 FORMAT (2F20.5)
      GO TO (6, 7, 8, 9, 10, 11, 12), II
    6 WRITE (6,13)
   13 FORMAT (1H0, 27H CLAMPED FREE END CONDITION)
      GO TO 20
    7 WRITE (6,14)
   14 FORMAT (1H0, 29H CLAMPED SIMPLE END CONDITION)
      GO TO 20
    8 WRITE(6,15)
   15 FORMAT (1H0, 29H CLAMPED PINNED END CONDITION)
      GO TO 20
    9 WRITE (6,16)
   16 FORMAT (1H0, 30H CLAMPED CLAMPED END CONDITION)
      GO TO 20
   10 WRITE (6,17)
   17 FORMAT (1H0, 26H PINNED FREE END CONDITION)
      GO TO 20
   11 WRITE (6,18)
   18 FORMAT (1H0, 28H PINNED SIMPLE END CONDITION)
      GO TO 20
   12 WRITE (6,19)
   19 FORMAT (1H0, 28H PINNED PINNED END CONDITION)
   20 WRITE (6,226)
  226 FORMAT (1H0, 33H ROTATORY INERTIA IS NOT INCLUDED)
      CALL MAPRIN (ROOT, 1, LIM, 1, 2, FREQ)
C-----
C-----NORMALLIZING OF X AND Y DISPLACEMENTS
C-----
      DO 56 I=1,LIM
      SUM = 0.0
      DO 52 J=1,N
   52 SUM = SUM + Y(I,J)**2 + X(I,J)**2
   55 SUM = DSQRT (SUM)
      DO 56 J=1,N
      Y(I,J) = Y(I,J) / SUM
   56 X(I,J) = X(I,J) / SUM
      CALL MAPRIN (X, LIM, N, 10, 6, XDISP)
      CALL MAPRIN (Y, LIM, N, 10, 6, YDISP)
   99 CONTINUE
      STOP
      END
      SUBROUTINE MAPRIN (A,N,M,ND,ICODE,TITLE)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION A(ND,1), TITLE(4), NKOL(6), F(4,6), FM1(2), FM2(2),
     1 F1(4), F2(4), F3(4), F4(4), F5(4), F6(4)
      EQUIVALENCE (F1,F),(F2,F(5)),(F3,F(9)),(F4,F(13)),(F5,F(17)),
     1 (F6,F(21))
C
C     MAPRIN PRINTS 'N' ROWS AND 'M' COLUMNS OF MATRIX A (ROW
C     DIMENSION = ND) BY PARTITIONING INTO GROUPS OF 'NK' COLUMNS.
C     ICODE = 1 TO 6 SPECIFIES THE FORMAT (SEE DATA STATEMENT FOR F).
C     THIS VERSION FOR IBM 360.
C
      DATA NKOL /8,8,10,10,12,12/
      DATA  F1 /32H(/2X 8I15)       (I5,2X, 8D15.7)/,
     1      F2 /32H(/2X 8I15)       (I5,1X, 8F15.3)/,
     2      F3 /32H(/3X10I12)       (I5,2X,10D12.4)/,
     3      F4 /32H(/3X10I12)       (I5,1X,10F12.6)/,
     4      F5 /32H(/4X12I10)       (I5,2X,12D10.2)/,
     5      F6 /32H(/4X12I10)       (I5,1X,12F10.4)/
      IF (ICODE.LT.1.OR.ICODE.GT.6)  ICODE = 1
      NK = NKOL(ICODE)
      FM1(1) = F(1,ICODE)
      FM1(2) = F(2,ICODE)
      FM2(1) = F(3,ICODE)
      FM2(2) = F(4,ICODE)
      NPR = 1 + (M-1)/NK
      N1 = 1
      N2 = MINO(M,NK)
      PRINT 10, TITLE
   10 FORMAT (/1H0,4A8)
      DO 150  K = 1,NPR
      PRINT FM1, (J, J=N1,N2)
      DO 140  I = 1,N
  140 PRINT FM2, I, (A(I,J), J=N1,N2)
      N1 = N1 + NK
  150 N2 = MINO(M,N2+NK)
      RETURN
      END
      SUBROUTINE EVAL (OMEGA, N, II, DET)
C-----
C-----SUBROUTINE GENERATES TRANSFER MATRICES AND EVALUATES FREQUENCY DET
C-----
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION U(6,108), SECL(20), G(6,108), VEC(6,6), VECA(6,3),
     1          VECB(6,3), XIX(20), E(20), WL(20)
      EQUIVALENCE (VEC(1,1), VECA(1,1)), (VEC(1,4), VECB(1,1))
      COMMON U, SECL, VEC, G, XIX, E, WL, KTROL
      NN = N - 1
C-----
C-----TRANSFER MATRIX GENERATION SECTION
C-----
      JN = 1
      DO 6  I=1,NN
      BETA4 = (OMEGA**2) * WL(I) * (SECL(I)**4) / E(I) / XIX(I)
      BETA2 = DSQRT (BETA4)
      BETA1 = DSQRT (BETA2)
      C0 = 0.5D0 * (DCOSH(BETA1) + DCOS(BETA1))
      C1 = 0.5D0 * (DSINH(BETA1) + DSIN(BETA1)) / BETA1
      C2 = 0.5D0 * (DCOSH(BETA1) - DCOS(BETA1)) / BETA2
      C3 = 0.5D0 / (BETA1**3) * (DSINH(BETA1) - DSIN(BETA1))
      Q = SECL(I)**2  / E(I) / XIX(I)
      U(1,JN) = 0.10D1
      U(2,JN) = 0.0
      U(3,JN) = 0.0
      U(4,JN) = 0.0
      U(5,JN) = 0.0
      U(6,JN) = - WL(I) * SECL(I) * (OMEGA**2)
      U(1,JN+1) = 0.0
      U(2,JN+1) = C0
      U(3,JN+1) = BETA4 * C3 / SECL(I)
      U(4,JN+1) = BETA4 * C2 / Q
      U(5,JN+1) =-BETA4 * C1 / Q / SECL(I)
      U(6,JN+1) = 0.0
      U(1,JN+2) = 0.0
      U(2,JN+2) = C1 * SECL(I)
      U(3,JN+2) = C0
      U(4,JN+2) = BETA4 * SECL(I) * C3 / Q
      U(5,JN+2) =-BETA4 * C2 / Q
      U(6,JN+2) = 0.0
      U(1,JN+3) = 0.0
      U(2,JN+3) = Q * C2
      U(3,JN+3) = Q * C1 / SECL(I)
      U(4,JN+3) = C0
      U(5,JN+3) =-BETA4 * C3 / SECL(I)
      U(6,JN+3) = 0.0
      U(1,JN+4) = 0.0
      U(2,JN+4) =-Q * C3 * SECL(I)
      U(3,JN+4) =-Q * C2
      U(4,JN+4) =-SECL(I) * C1
      U(5,JN+4) = C0
      U(6,JN+4) = 0.0
      U(1,JN+5) = 0.0
      U(2,JN+5) = 0.0
      U(3,JN+5) = 0.0
      U(4,JN+5) = 0.0
      U(5,JN+5) = 0.0
      U(6,JN+5) = 0.1D01
    6 JN = JN + 6
      JJ = 6 * NN
      DO 7 I=1,6
      DO 7 J=1,3
    7 VEC(I,J) = 0.0
C-----
C-----LEFT END CONDITIONS
C-----
      GO TO (10, 10, 10, 10, 11, 11, 11), II
   10 VEC(4,1) = 0.1D1
      GO TO 8
   11 VEC(3,1) = 0.1D1
    8 VEC(5,2) = 0.1D1
      VEC(6,3) = 0.1D1
      IF (KTROL.GT.1) VEC(6,3) = 0.0
   27 JN = 0
      DO 9 L=1,NN
C-----
C-----PASSING END CONDITIONS THROUGH TRANSFER MATRICES AND ROTATIONS
C-----
      DO 12 K=1,3
      DO 12 I=1,6
      VECB(I,K) = 0.0
      DO 12 J=1,6
   12 VECB(I,K) = VECB(I,K) + G(I,JN+J) * VECA(J,K)
      DO 13 K=1,3
      DO 13 I=1,6
      VECA(I,K) = 0.0
      DO 13 J=1,6
   13 VECA(I,K) = VECA(I,K) + U(I,JN+J) * VECB(J,K)
    9 JN = JN + 6
```

```
      DO 14 K=1,3
      DO 14 I=1,6
      VECB(I,K) = 0.0
      DO 14 J=1,6
   14 VECB(I,K) = VECB(I,K) + G(I,JN+J) * VECA(J,K)
C-----
C-----RIGHT END CONDITIONS
C-----
      GO TO (15, 16, 17, 18, 15, 16, 17), II
   15 L1 = 4
      L2 = 5
      L3 = 6
      GO TO 19
   16 L1 = 2
      L2 = 4
      L3 = 6
      GO TO 19
   17 L1 = 1
      L2 = 2
      L3 = 4
      GO TO 19
   18 L1 = 1
      L2 = 2
      L3 = 3
      GO TO 19
C-----
C-----EVALUATION OF FREQUENCY DETERMINANT
C-----
   19 DO 20 I=1,3
      VECA(1,I) = VECB(L1,I)
      VECA(2,I) = VECB(L2,I)
   20 VECA(3,I) = VECB(L3,I)
      IF (KTROL.GT.1) GO TO 22
   23 DET = VECA(1,1) * (VECA(2,2) * VECA(3,3) - VECA(3,2) * VECA(2,3))
     1    - VECA(1,2) * (VECA(2,1) * VECA(3,3) - VECA(3,1) * VECA(2,3))
     2    + VECA(1,3) * (VECA(2,1) * VECA(3,2) - VECA(3,1) * VECA(2,2))
      RETURN
   22 GO TO (28, 28, 21, 21, 28, 28, 21), II
   21 DET = VEC(2,1) * VEC(3,2) - VEC(3,1) * VEC(2,2)
      RETURN
   28 DET = VEC(1,1) * VEC(2,2) - VEC(2,1) * VEC(1,2)
      RETURN
      END
```

```
TRANSFER MATRIX METHOO
      X COORD(IN)          Y COORD(IN)   MOOULUS OF ELASTICITY(PSI)   SPECIFIC WT(L8/CU IN)   AREA(SO IN)        MOMENT(IN 4)
          0.0                  0.0
                                          30000000.00000               0.28300                4.62000            30.30000
          0.0                 60.00000
                                          30000000.00000               0.28300                4.62000            30.30000
          0.0                120.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        180.00000            156.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        360.00000            192.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        540.00000            156.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        720.00000            120.00000
                                          30000000.00000               0.28300                5.88000            82.50000
        720.00000             60.00000
                                          30000000.00000               0.28300                5.88000            82.50000
        720.00000              0.0
CLAMPED CLAMPED ENO CONOITION
ROTATORY INERTIA IS NOT INCLUOEO

NATURAL FREQUENCIES(RAO/SEC)
            1              2               3               4              5
  1       25.754         34.680          79.630         150.284        279.474

NORMALIZEO X OISPLACEMENTS
            1          2          3          4          5          6          7          8          9
  1       0.0       0.2051     0.3679     0.3015     0.2577     0.2311     0.1475     0.0364    -0.0000
  2       0.0      -0.0932    -0.2686    -0.3468    -0.3441    -0.3988    -0.4196    -0.1981    -0.0000
  3       0.0       0.1616     0.1678     0.0422     0.1813     0.0614     0.1948     0.1604    -0.0000
  4       0.0       0.0551    -0.1330    -0.2284    -0.0127     0.2129     0.1077    -0.0587    -0.0000
  5       0.0       0.7538     0.0479     0.0435     0.0352    -0.0149     0.0225     0.6213     0.0000

NORMALIZEO Y DISPLACEMENTS
            1          2          3          4          5          6          7          8          9
  1       0.0      -0.0000    -0.0000     0.3318     0.5511     0.4183    -0.0000    -0.0000    -0.0000
  2       0.0       0.0000     0.0000     0.3911     0.3775     0.1036    -0.0000    -0.0000    -0.0000
  3       0.0      -0.0000    -0.0000     0.6279    -0.0674    -0.6670    -0.0000    -0.0000    -0.0000
  4       0.0      -0.0000     0.0000     0.4770    -0.6017     0.5262    -0.0000    -0.0000    -0.0000
  5       0.0      -0.0000    -0.0000     0.0219     0.0633    -0.1874     0.0000     0.0000     0.0000
```

```
TRANSFER MATRIX METHOD
      X COORD(IN)          Y COORD(IN)   MOOULUS OF ELASTICITY(PSI)   SPECIFIC WT(L8/CU IN)   AREA(SO IN)        MOMENT(IN 4)
          0.0                  0.0
                                          30000000.00000               0.28300                4.62000            30.30000
          0.0                 60.00000
                                          30000000.00000               0.28300                4.62000            30.30000
          0.0                120.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        180.00000            156.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        360.00000            192.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        540.00000            156.00000
                                          30000000.00000               0.28300               29.40000           625.00000
        720.00000            120.00000
                                          30000000.00000               0.28300                5.88000            82.50000
        720.00000             60.00000
                                          30000000.00000               0.28300                5.88000            82.50000
        720.00000              0.0
PINNEO PINNEO ENO CONDITION
ROTATORY INERTIA IS NOT INCLUDEO

NATURAL FREOUENCIES(RAO/SEC)
            1              2               3               4              5
  1       15.257         25.365          75.652         149.662        277.069

NORMALIZEO X OISPLACEMENTS
            1          2          3          4          5          6          7          8          9
  1       0.0       0.2982     0.4299     0.4152     0.3960     0.3957     0.3622     0.2170    -0.0000
  2       0.0       0.0689     0.0054    -0.0927    -0.1257    -0.1777    -0.2569    -0.2132     0.0000
  3       0.0       0.2040     0.1145    -0.0153     0.1233    -0.0020     0.1320     0.1993    -0.0000
  4       0.0       0.0993    -0.1289    -0.2238    -0.0078     0.2159     0.1133    -0.0961    -0.0000
  5       0.0       0.7861     0.0052     0.0061    -0.0013    -0.0229    -0.0077     0.6120    -0.0001

NORMALIZEO Y OISPLACEMENTS
            1          2          3          4          5          6          7          8          9
  1       0.0      -0.0000    -0.0000     0.0733     0.1693     0.1678    -0.0000    -0.0000    -0.0000
  2       0.0      -0.0000    -0.0000     0.4906     0.6559     0.3961     0.0000     0.0000     0.0000
  3       0.0      -0.0000    -0.0000     0.6493    -0.0438    -0.6702    -0.0000    -0.0000    -0.0000
  4       0.0      -0.0000     0.0000     0.4743    -0.6055     0.5130    -0.0000    -0.0000    -0.0000
  5       0.0      -0.0000    -0.0000    -0.0046     0.0323    -0.0758    -0.0000    -0.0000    -0.0000
```

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13. ABSTRACT

Lumped-parameter and finite-element-models are developed for the dynamic behavior of plane straight sided frames. The models do not include axial and shear deformation, but the equations of motion developed using the models allow for time varying external loading. The performance of these models is evaluated by a comparison with a standard transfer matrix method for the special case of free undamped vibration. The finite-element-model proves to be much the better model. For the first five modes, the finite-element-model with 27 degrees of freedom differs by no more than 0.4% from the "exact values" given by the transfer matrix method. The lumped-parameter-model with 39 degrees of freedom gives errors roughly ten times as great.

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VIBRATIONS OF FRAMES						
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